Introduction to Machine Learning

Convolutional Networks

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August 22, 2020

Outline

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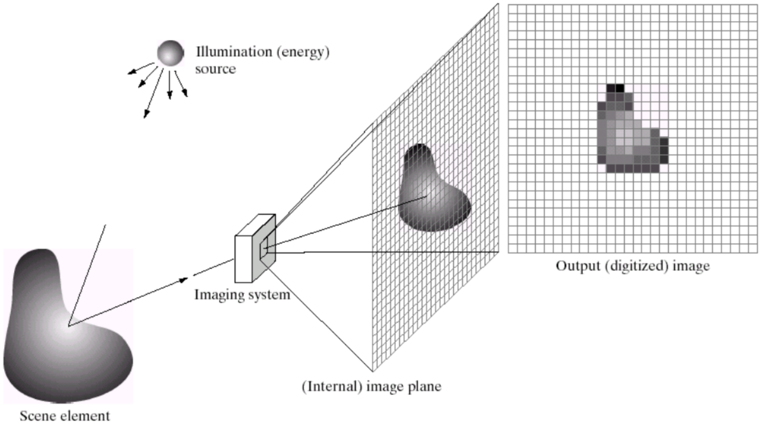
Digital Images as pixels in a digitized matrix [1]



Further [1]

Pixel values typically represent

- Gray levels, colors, heights, opacities etc

Something Notable

- Remember digitization implies that a digital image is an approximation of a real scene

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Images

Common image formats include

- On sample/pixel per point (B&W or Grayscale)
- Three samples/pixel per point (Red, Green, and Blue)
- Four samples/pixel per point (Red, Green, Blue, and "Alpha")

Therefore, we have the following process

Low Level Process

Input	Processes	Output
Image	Noise Removal	Improved Image
	Image Sharpening	

Example

Edge Detection

Example

Edge Detection

Then

Mid Level Process

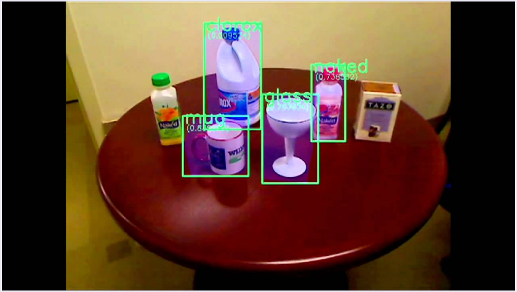
Input	Processes	Output
Image	Object Recognition Segmentation	Attributes

Example

Object Recognition

Example

Object Recognition



Therefore

It would be nice to automatize all these processes

- We would solve a lot of headaches when setting up such process

Why not to use the data sets

- By using a Neural Networks that replicates the process.

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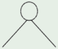
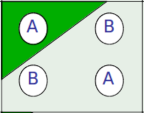
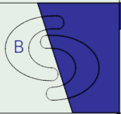
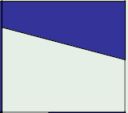
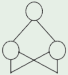
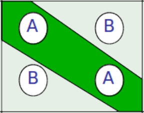
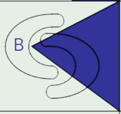
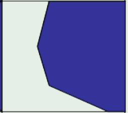
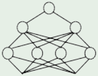
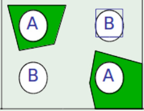
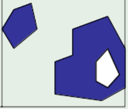
Multilayer Neural Network Classification

We have the following classification [2]

Structure	Types of Decision Regions	Exclusive-OR Problem	Classes with Meshed regions	Most General Region Shapes
Single-Layer 	Half Plane Bounded By Hyper plane			
Two-Layer 	Convex Open Or Closed Regions			
Three-Layer 	Arbitrary (Complexity Limited by No. of Nodes)			

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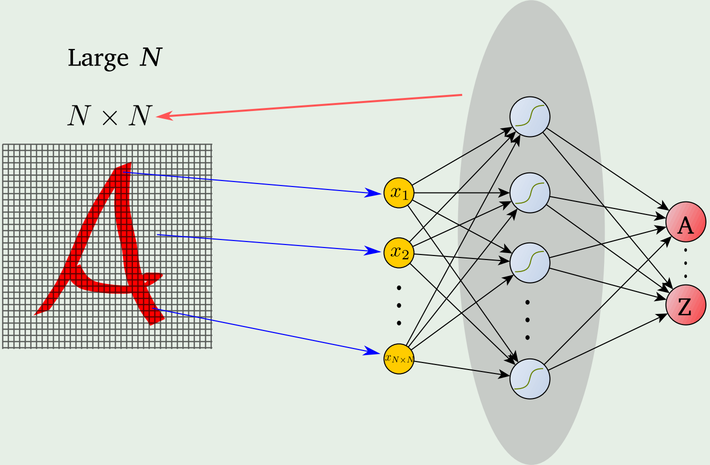
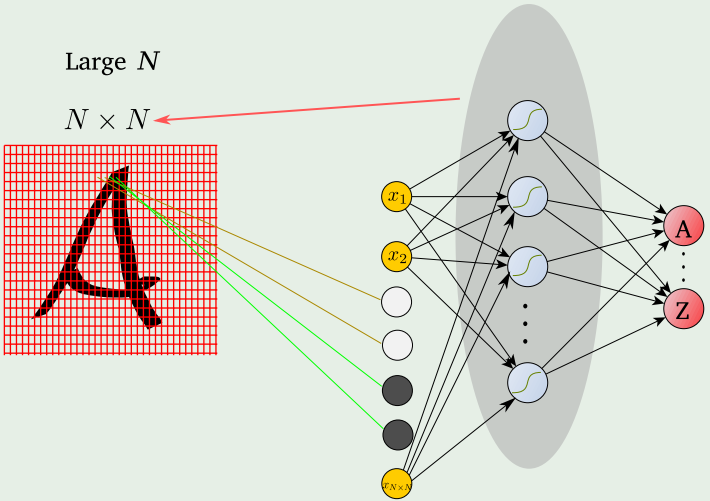
Drawbacks of previous neural networks

The number of trainable parameters becomes extremely large



Drawbacks of previous neural networks

In addition, little or no invariance to shifting, scaling, and other forms of distortion



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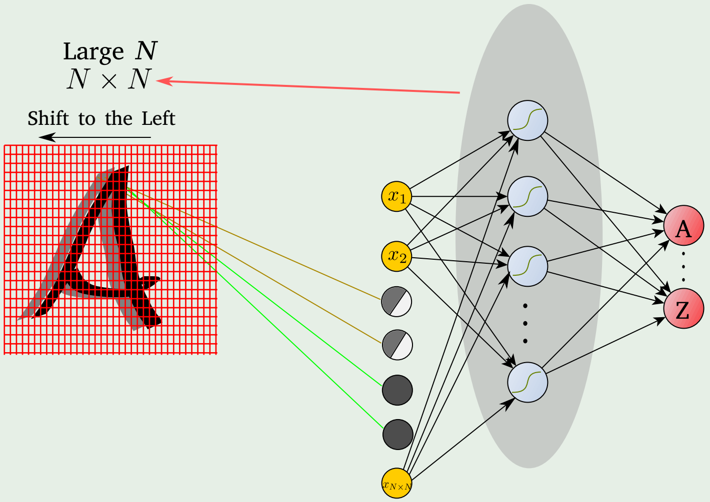
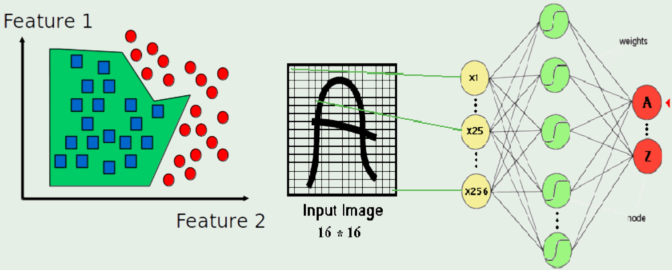


Drawbacks of previous neural networks

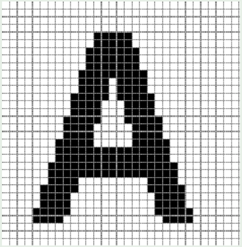
The topology of the input data is completely ignored



For Example

We have

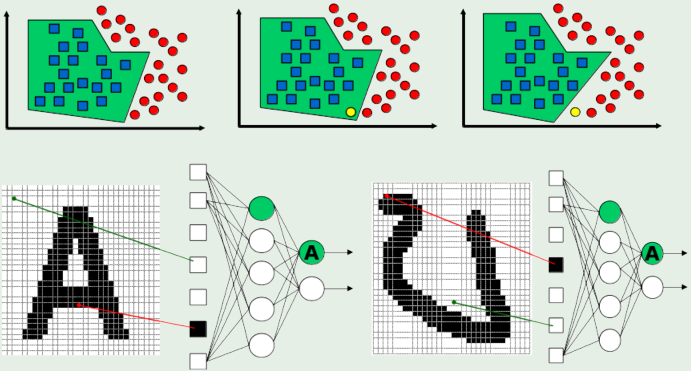
- Black and white patterns: $2^{32\times 32} = 2^{1024}$
- Gray scale patterns: $256^{32\times 32} = 256^{1024}$



32 * 32 input image

For Example

If we have an element that the network has never seen



Possible Solution

We can minimize this drawbacks by getting

Fully connected network of sufficient size can produce outputs that are invariant with respect to such variations.

Problem!!!

- Training time
- Network size
- Free parameters

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Something Notable [3]

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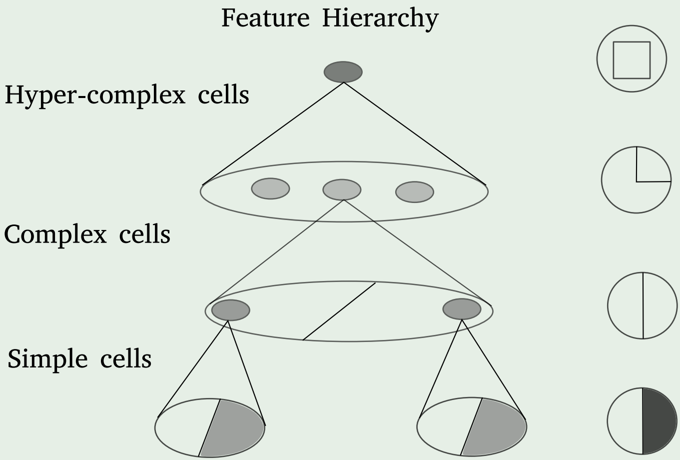
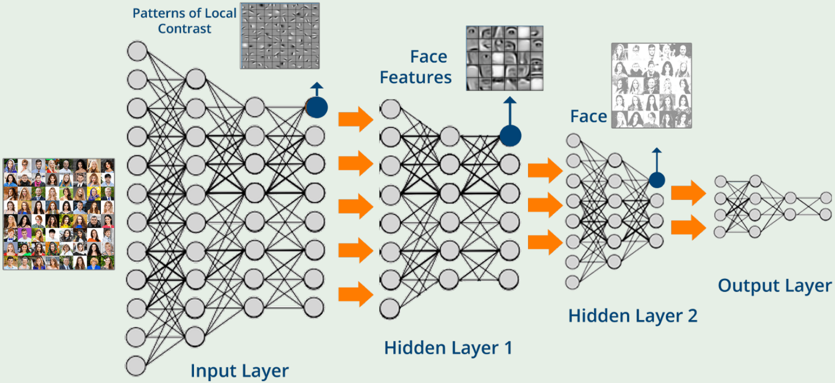


History

Convolutional Neural Networks (CNN) were invented by [4]

In 1989, Yann LeCun and Yoshua Bengio introduced the concept of Convolutional Neural networks.



About CNN's

Something Notable

CNN's Were neurobiologically motivated by the findings of locally sensitive and orientation-selective nerve cells in the visual cortex.

In addition

They designed a network structure that implicitly extracts relevant features.

Properties

Convolutional Neural Networks are a special kind of multilayer neural networks.

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In addition

- CNN is a feed-forward network that can extract topological properties from an image.
- Like almost every other neural networks they are trained with a version of the back-propagation algorithm.
- Convolutional Neural Networks are designed to recognize visual patterns directly from pixel images with minimal preprocessing.
- They can recognize patterns with extreme variability.

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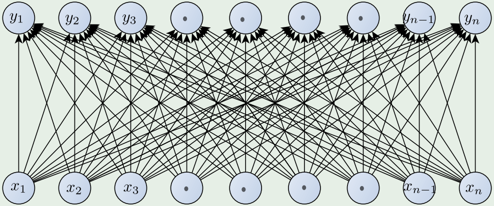
Local Connectivity

We have the following idea [5]

- Instead of using a full connectivity...



Input Image

We would have something like this

$$y_i = f\left(\sum_{i=1}^{n} w_i x_i\right) \quad (1)$$

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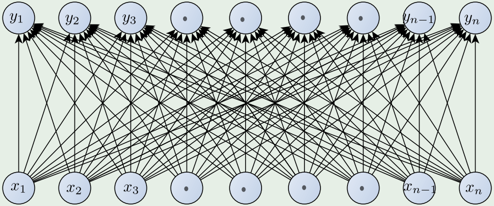

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Local Connectivity

We decide only to connect the neurons in a local way

- Each hidden unit is connected only to a subregion (patch) of the input image.
- It is connected to all channels:
  - 1 if gray scale
  - 3 in the RGB case

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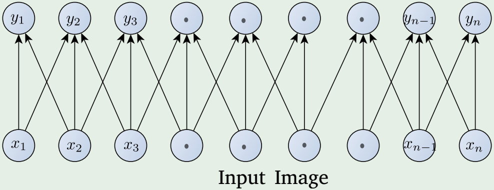
For gray scale, we get something like this



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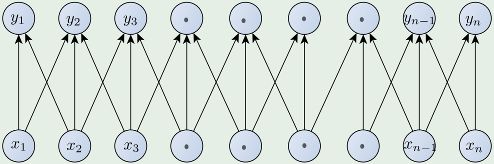

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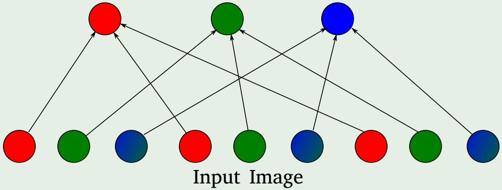
Example

In the case of the 3 channels



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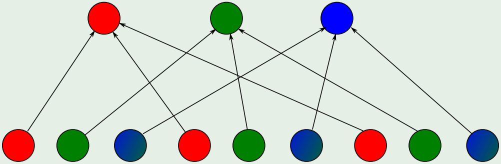

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How this looks in the image...

We have

Receptive Field

Outline

Parameter Sharing

Second Idea

Share matrix of parameters across certain units.

These units are organized into

- The same feature "map"
  - Where the units share same parameters (For example, the same mask)

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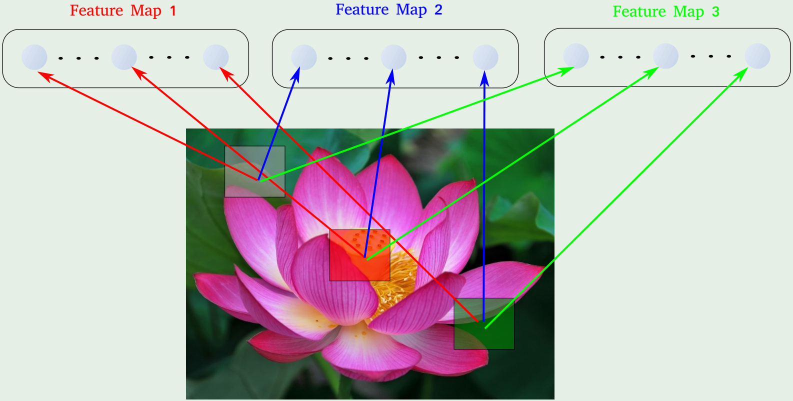

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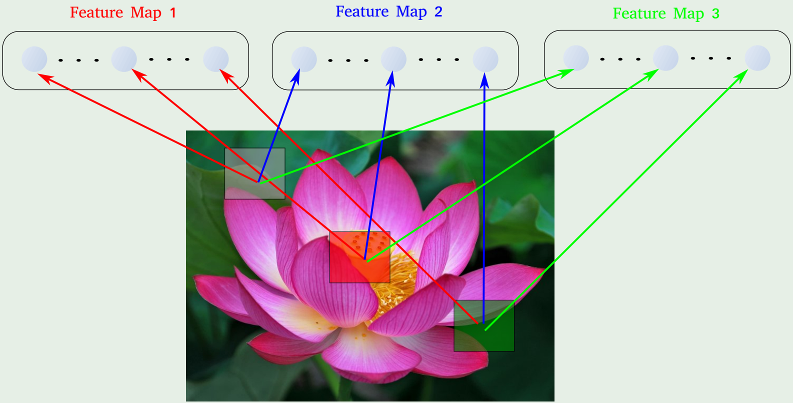

Now, in our notation

We have a collection of matrices representing this connectivity

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An now why the name of convolution

Yes!!! The definition is coming now.

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In computer vision [1, 6]

We usually operate on digital (discrete) images:

- Sample the 2D space on a regular grid.
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The image can now be represented as a matrix of integer values, $f : [a, b] \times [c, d] \rightarrow I$

$$\begin{array}{cc} & j \longrightarrow \\ i \downarrow & \begin{bmatrix} 79 & 5 & 6 & 90 & 12 & 34 & 2 & 1 \\ 8 & 90 & 12 & 34 & 26 & 78 & 34 & 5 \\ 8 & 1 & 3 & 90 & 12 & 34 & 11 & 61 \\ 77 & 90 & 12 & 34 & 200 & 2 & 9 & 45 \\ 1 & 3 & 90 & 12 & 20 & 1 & 6 & 23 \end{bmatrix} \end{array}$$

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We can see the coordinate of f as follows

We have the following

$$f = \begin{pmatrix} f_{-n,-n} & f_{-n,-n+1} & \cdots & f_{-n,(n-1)} & f_{-n,n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ \vdots & \cdots & f_{0,0} & \cdots & \vdots \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ f_{n\times -n} & f_{n\times -n+1} & \cdots & f_{n\times(n-1)} & f_{n,n} \end{pmatrix} \tag{4}$$

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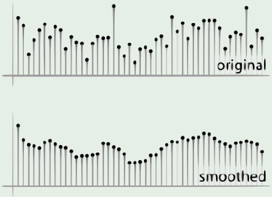
Many times we want to eliminate noise in a image

By using for example a moving average



This last moving average can be seen as

$$(f * g)(i) = \sum_{j=-n}^{n} f(j)\, g(i-j) = \frac{1}{N} \sum_{j=m}^{-m} f(j) \quad (5)$$

With $f(j)$ representing the value of the pixel at position i,

$$g(h) = \begin{cases} \frac{1}{N} & \text{if } h \in \{-m, -m+1, ..., 1, 0, 1, ..., m-1, m\} \\ 0 & \text{else} \end{cases}$$

with $0 < m < n$.

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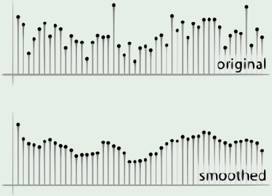

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This can be generalized into the 2D images

Left f and Right $f * g$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0								

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0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0	10							

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0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0	10	20						

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0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Moving average in 2D

Basically in 2D

We have that we can define different types of filter using the idea of weighted average

$$(f * g)(i,j) = \sum_{k=n}^{-n} \sum_{l=-n}^{n} f(k,l) \times g(i-k, j-l) \tag{6}$$

What is this weight matrix also called a kernel of 3×3 moving average

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{"The Box Filter"} \tag{7}$$

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Outline

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Convolution

Definition

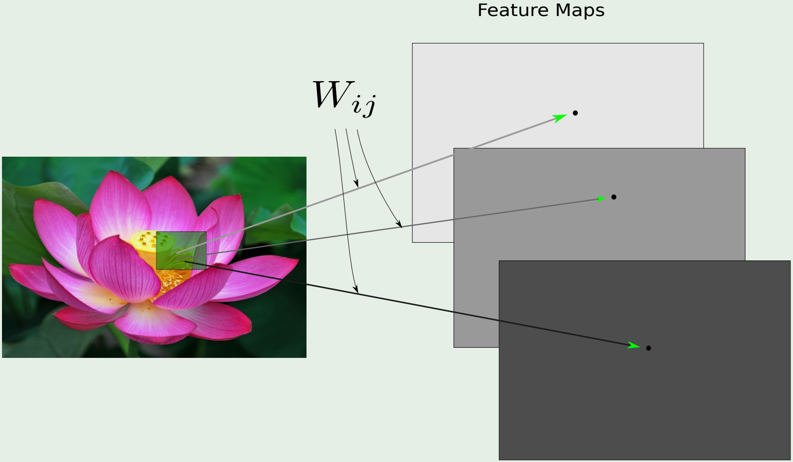
Let $f : [a, b] \times [c, d] \to I$ be the image and $g : [e, f] \times [h, i] \to V$ be the kernel. The output of convolving f with g, denoted $f * g$ is

$$(f * g)[x, y] = \sum_{k=-n}^{n} \sum_{l=-n}^{n} f(k, l)\, g(x - k, y - l) \tag{8}$$

- The Flipped Kernel

Back on the Convolutional Architecture

We have then something like this



Thus

Each Feature Map forms a 2D grid of features

That can be computed with a discrete convolution (*) of a kernel matrix k_{ij} which is the hidden weights matrix W_{ij} with rows and columns with its rows and columns flipped.

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In addition

- x_i is the ith channel of input.
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$$y_j = \sum_i k_{ij} * x_i \tag{9}$$

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Furthermore

Let layer l be a Convolutional Layer

Then, the input of layer l comprises $m_1^{(l-1)}$ feature maps from the previous layer.

Each input layer has a size of $m_2^{(l-1)} \times m_3^{(l-1)}$

In the case where $l = 1$, the input is a single image I consisting of one or more channels.

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Remark

We have that

- A Convolutional Neural Network (CNN) directly accepts raw images as input.

Thus, their importance when training discrete filters

- Instead of assuming a certain comprehension of Computer Vision, one could think this is as a Silver Bullet.

However, you still

- You still need to be aware of :
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Another Remark

We have the following

- $Y_j^{(l)}$ is a matrix representing the l layer and j^{th} feature map.

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- We can see the convolutional as a fusion of information from different feature maps.

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Given a specific layer l, we have that i^{th} feature map in such layer equal to

$$Y_i^{(l)} = B_i^{(l)} + \sum_{j=1}^{m_1^{(l-1)}} K_{ij}^{(l)} * Y_j^{(l-1)} \tag{10}$$

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- $Y_i^{(l)}$ is the i^{th} feature map in layer l.
- $B_i^{(l)}$ is the bias matrix for output j.
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Something Notable

- $m_2^{(l)}$ and $m_3^{(l)}$ are influenced by border effects.
- Therefore, the output feature maps when the convolutional sum is defined properly have size

$$m_2^{(l)} = m_2^{(l-1)} - 2h_1^{(l)}$$

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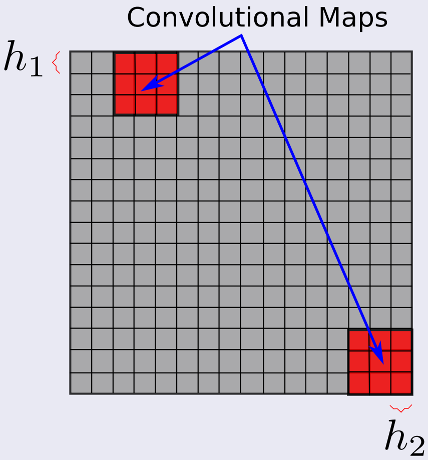
Why?

Example



Special Case

When $l = 1$

The input is a single image I consisting of one or more channels.

Thus

We have

Each feature map $Y_i^{(l)}$ in layer l consists of $m_1^{(l)} \cdot m_2^{(l)}$ units arranged in a two dimensional array.

Thus, the unit at position (r, s) computes

$$
\begin{aligned}
\left(Y_i^{(l)}\right)_{r,s} &= \left(B_i^{(l)}\right)_{r,s} + \sum_{j=1}^{m_1^{(l-1)}} \left(K_{ij}^{(l)} * Y_j^{(l-1)}\right)_{r,s} \\
&= \left(B_i^{(l)}\right)_{r,s} + \sum_{j=1}^{m_1^{(l-1)}} \sum_{k=-h_1^{(l)}}^{h_1^{(l)}} \sum_{t=-h_2^{(l)}}^{h_2^{(l)}} \left(K_{ij}^{(l)}\right)_{k,t} \left(Y_j^{(l-1)}\right)_{r+k,s+t}
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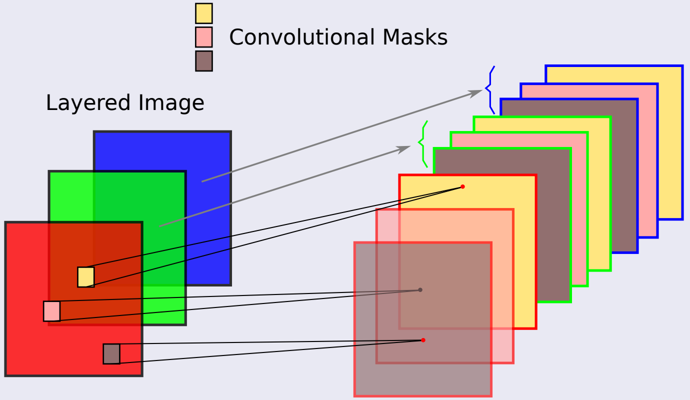
Example

A Convolutional Layer against a RGB Image using three masks/filters



Outline

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As in Multilayer Perceptron

We use a non-linearity

- However, there is a drawback when using Back-Propagation under a sigmoid function

$$s(x) = \frac{1}{1+e^{-x}}$$

Because if we imagine a Convolutional Network as a series of layer functions f_i

$$y(A) = f_t \circ f_{t-1} \circ \cdots \circ f_2 \circ f_1(A)$$

With f_t is the last layer.

Therefore, we finish with a sequence of derivatives

$$\frac{\partial y(A)}{\partial w_{1i}} = \frac{\partial f_t(f_{t-1})}{\partial f_{t-1}} \cdot \frac{\partial f_{t-1}(f_{t-2})}{\partial f_{t-2}} \cdot \cdots \cdot \frac{\partial f_2(f_1)}{\partial f_2} \cdot \frac{\partial f_1(A)}{\partial w_{1i}}$$

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Given the commutativity of the product

- You could put together the derivative of the sigmoid's

$$f(x) = \frac{ds(x)}{dx} = \frac{e^{-x}}{\left(1 + e^{-x}\right)^2}$$

Therefore, deriving again

$$\frac{df(x)}{dx} = -\frac{e^{-x}}{\left(1 + e^{-x}\right)^2} + \frac{2\left(e^{-x}\right)^2}{\left(1 + e^{-x}\right)^3}$$

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The need to introduce a new function

$$f(x) = x^{+} = \max(0, x)$$

It is called ReLu or Rectifier

With a smooth approximation (Softplus function)

$$f(x) = \frac{\ln\left(1 + e^{kx}\right)}{k}$$

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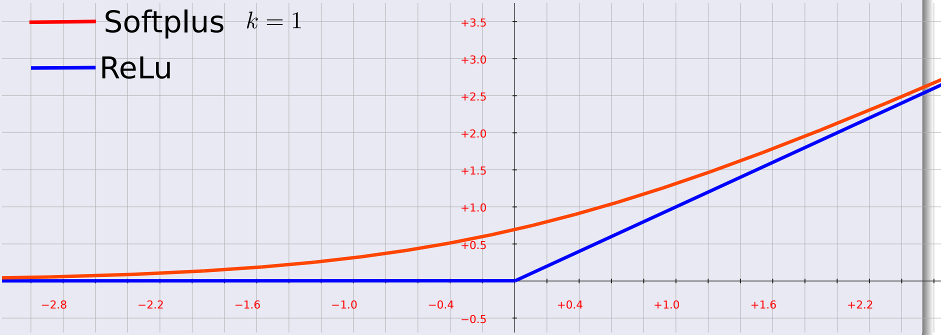
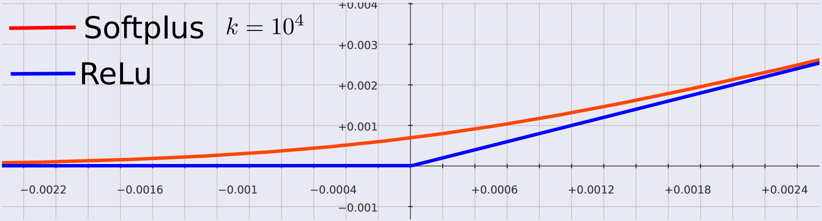


Increase k

When $k = 10^4$



Non-Linearity Layer

If layer l is a non-linearity layer

Its input is given by $m_1^{(l)}$ feature maps.

What about the output

Its output comprises again $m_1^{(l)} = m_1^{(l-1)}$ feature maps

Each of them of size

$$m_2^{(l-1)} \times m_3^{(l-1)} \tag{11}$$

With $m_2^{(l)} = m_2^{(l-1)}$ and $m_3^{(l)} = m_3^{(l-1)}$.

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With the final output

$$Y_i^{(l)} = f\left(Y_i^{(l-1)}\right) \quad (12)$$

Where

f is the activation function used in layer l and operates point wise.

You can also add a gain

$$Y_i^{(l)} = g_i f\left(Y_i^{(l-1)}\right) \quad (13)$$

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Outline

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Rectification Layer, R_{abs}

Now a rectification layer

Then its input comprises $m_1^{(l)}$ feature maps of size $m_2^{(l-1)} \times m_3^{(l-1)}$.

Then, the absolute value for each component of the feature maps is computed

$$Y_i^{(l)} = \left| Y_i^{(l)} \right| \tag{14}$$

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It is computed point wise such that the output consists of $m_1^{(l)} = m_1^{(l-1)}$ feature maps unchanged in size.

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$$f(x) = \frac{\ln\left(1 + e^{kx}\right)}{k}$$

We have that

Experiments show that rectification plays a central role in achieving good performance.

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Remark

- Rectification could be included in the non-linearity layer.
- But also it can be seen as an independent layer.

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Given that we are using Backpropagation

We need a soft approximation to $f(x) = |x|$

For this, we have

$$\frac{\partial f}{\partial x} = \mathsf{sgn}(x)$$

- When $x \neq 0$. Why?

We can use the following approximation

$$\mathsf{sgn}(x) = 2\left(\frac{\exp\{kx\}}{1+\exp\{kx\}}\right) - 1$$

Therefore, we have by integration and working the C

$$f(x) = \frac{2}{k}\ln(1+\exp\{kx\}) - x - \frac{2}{k}\ln(2)$$

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We get the following situation

Something Notable

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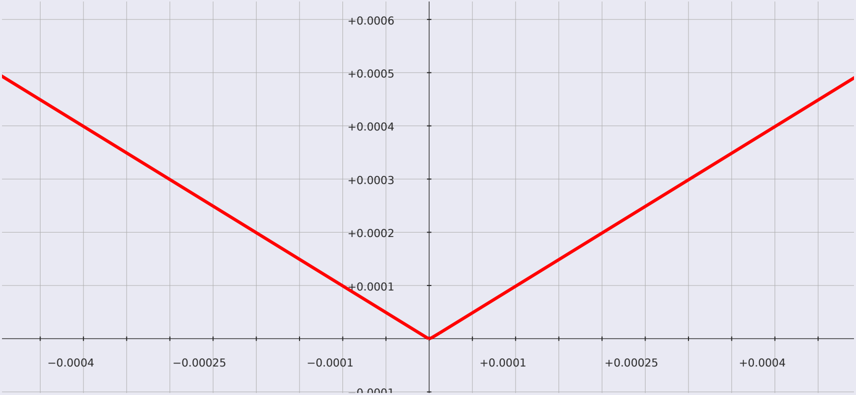

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Outline

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Normalizing

Contrast normalization layer

The task of a local contrast normalization layer:

- To enforce local competitiveness between adjacent units within a feature map.
- To enforce competitiveness units at the same spatial location.

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Normalizing

Contrast normalization layer

The task of a local contrast normalization layer:

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- To enforce competitiveness units at the same spatial location.

We have two types of operations

- Subtractive Normalization.
- Brightness Normalization.

Normalizing

Contrast normalization layer

The task of a local contrast normalization layer:

- To enforce local competitiveness between adjacent units within a feature map.
- To enforce competitiveness units at the same spatial location.

We have two types of operations

- Subtractive Normalization.
- Brightness Normalization.

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Subtractive Normalization

Given $m_1^{(l-1)}$ feature maps of size $m_2^{(l-1)} \times m_3^{(l-1)}$

The output of layer l comprises $m_1^{(l)} = m_1^{(l-1)}$ feature maps unchanged in size.

With the operation

$$Y_i^{(l)} = Y_i^{(l-1)} - \sum_{j=1}^{m_1^{(l-1)}} K_{G(\sigma)} * Y_j^{(l-1)} \tag{15}$$

With

$$\left(K_{G(\sigma)}\right)_{r,s} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{r^2+s^2}{2\sigma^2}\right\} \tag{16}$$

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Brightness Normalization

An alternative is to normalize the brightness in combination with the rectified linear units

$$\left(Y_i^{(l)}\right)_{r,s} = \frac{\left(Y_i^{(l-1)}\right)_{r,s}}{\left(\kappa + \lambda \sum_{j=1}^{m_1^{(l-1)}} \left(Y_j^{(l-1)}\right)_{r,s}^2\right)^{\mu}} \quad (17)$$

Where

- κ, μ and λ are hyperparameters which can be set using a

$$f(x) = \frac{\ln\left(1 + e^{kx}\right)}{k}$$

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Outline

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Subsampling Layer

Motivation

The motivation of subsampling the feature maps obtained by previous layers is robustness to noise and distortions.

How?

- Normally, in traditional Convolutional Networks subsampling this is done by applying skipping factors!!!
- However, it is possible to combine subsampling with pooling and do it in a separate laye

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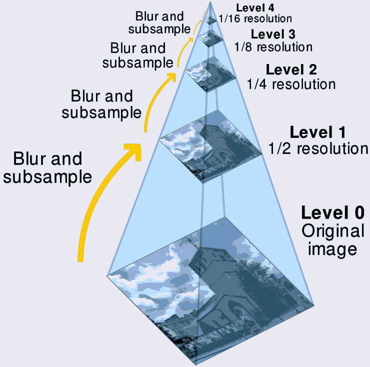
Sub-sampling

The subsampling layer

- It seems to be acting as the well know sub-sampling pyramid



How is subsampling implemented?

We know that Image Pyramids

They were designed to find:

1. filter-based representations to decompose images into information at multiple scales,
2. To extract features/structures of interest,
3. To attenuate noise.

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- The SURF and SIFT filters

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Projection Vectors

Let $I \in \mathbb{R}^N$ an image

And a projection transformation such that

$$\boldsymbol{a} = PI$$

Where

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_0 & \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{M-1} \end{bmatrix} \in \mathbb{R}^M$$

- The transformation coefficients...

Additionally, we have the projection vectors in P

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Thus, we have the following cases

When $M = N$

- Thus, the projection P is to be critically sampled (Relation with the rank of P)

When $N < M$

- Over-sampled

When $M < N$

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Therefore

We have that we can build a series of subsampled images

$$\left\{ \begin{matrix} I_0 & I_1 & \cdots & I_T \end{matrix} \right\}$$

Usually constructed with a separable 1D kernel h

$$I_{k+1} = PI_k = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\text{down-sampling}} \underbrace{\begin{pmatrix} \ddots & & & & & \\ - & h & - & & & \\ & - & h & - & & \\ & & - & h & - & \\ & & & & & \ddots \end{pmatrix}}_{\text{conv toplitz matrix}} I_k$$

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There are also other ways of doing this

subsampling can be done using so called skipping factors

$$s_1^{(l)} \text{ and } s_2^{(l)}$$

The basic idea is to skip a fixed number of pixels

Therefore the size of the output feature map is given by

$$m_2^{(l)} = \frac{m_2^{(l-1)} - 2h_1^{(l)}}{s_1^{(l)} + 1} \text{ and } m_3^{(l)} = \frac{m_3^{(l-1)} - 2h_2^{(l)}}{s_2^{(l)} + 1}$$

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What is Pooling?

Pooling

Spatial pooling is way to compute image representation based on encoded local features.

Pooling

Let l be a pooling layer

Its output comprises $m_1^{(l)} = m_1^{(l-1)}$ feature maps of reduced size.

Pooling Operation

It operates by placing windows at non-overlapping positions in each feature map and keeping one value per window such that the feature maps are subsampled.

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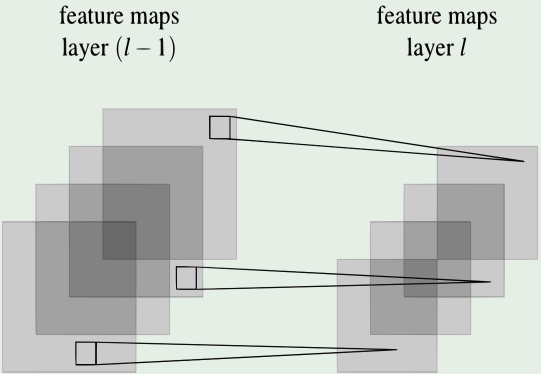
Example

If layer l is a pooling and subsampling layer and given $m_1^{(l-1)} = 4$ feature maps



Thus

In the previous example

All feature maps are pooled and subsampled individually.

Each unit

In one of the $m_1^{(l)} = 4$ output feature maps represents the average or the maximum within a fixed window of the corresponding feature map in layer $(l-1)$.

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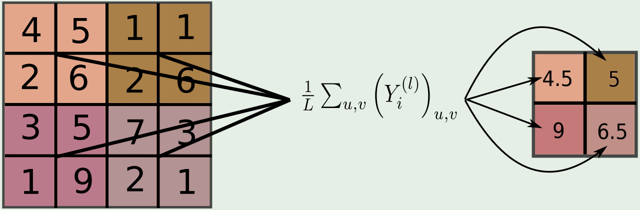
We distinguish two types of pooling

Average pooling

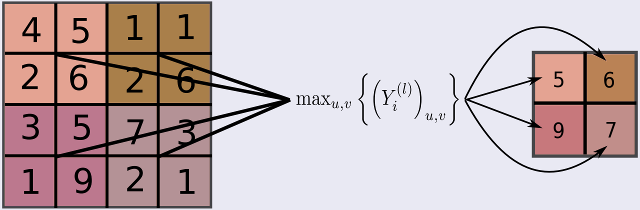
When using a boxcar filter, the operation is called average pooling and the layer denoted by P_A.



We distinguish two types of pooling

Max pooling

For max pooling, the maximum value of each window is taken. The layer is denoted by P_M.



Outline

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Fully Connected Layer

If a layer l is a fully connected layer

If layer $(l-1)$ is a fully connected layer, use the equation to compute the output of i^{th} unit at layer l:

$$z_i^{(l)} = \sum_{k=0}^{m^{(l)}} w_{i,k}^{(l)} y_k^{(l)} \text{ thus } y_i^{(l)} = f\left(z_i^{(l)}\right)$$

Otherwise

Layer l expects $m_1^{(l-1)}$ feature maps of size $m_2^{(l-1)} \times m_3^{(l-1)}$ as input.

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Then

Thus, the i^{th} unit in layer l computes

$$y_i^{(l)} = f\left(z_i^{(l)}\right)$$

$$z_i^{(l)} = \sum_{j=1}^{m_1^{(l-1)}} \sum_{r=1}^{m_2^{(l-1)}} \sum_{s=1}^{m_3^{(l-1)}} w_{i,j,r,s}^{(l)} \left(Y_j^{(l-1)}\right)_{r,s}$$

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Here

Where $w_{i,j,r,s}^{(l)}$

- It denotes the weight connecting the unit at position (r, s) in the j^{th} feature map of layer $(l-1)$ and the i^{th} unit in layer l.

Something Notable

- In practice, Convolutional Layers are used to learn a feature hierarchy and one or more fully connected layers are used for classification purposes based on the computed features.

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Basically

We can use a loss function at the output of such layer

$$L(\boldsymbol{W}) = \sum_{n=1}^{N} E_n(\boldsymbol{W}) = \sum_{n=1}^{N}\sum_{k=1}^{K}\left(y_{nk}^{(l)} - t_{nk}\right)^2 \text{ (Sum of Squared Error)}$$

$$L(\boldsymbol{W}) = \sum_{n=1}^{N} E_n(\boldsymbol{W}) = \sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk}\log\left(y_{nk}^{(l)}\right) \text{ (Cross-Entropy Error)}$$

Assuming W the tensor used to represent all the possible weights

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Outline

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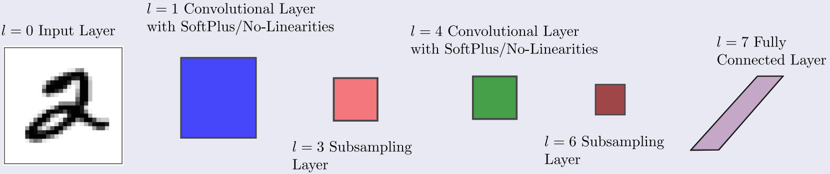
We have the following Architecture

Simplified Architecture by Jean LeCun "Backpropagation applied to handwritten zip code recognition"



Therefore, we have

Layer $l = 1$

- This Layer is using a Softplus f with 1 channels $j = 1$ Black and White

$$f\left[\left(Y_1^{(1)}\right)_{r,s}\right] = f\left[\left(B_1^{(l)}\right)_{r,s} + \sum_{k=-h_1^{(1)}}^{h_1^{(1)}} \sum_{t=-h_2^{(1)}}^{h_2^{(1)}} \left(K_{ij}^{(1)}\right)_{k,t} \left(Y_1^{(0)}\right)_{r+k,s+t}\right]$$

Now

We have the $l = 2$ subsampling for each coordinate

$$Y_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} f\left[\left(Y_1^{(1)}\right)\right] \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T$$

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Then, you repeat the previous

Thus we obtain a reduced convoluted version $Y_1^{(6)}$ of the $Y_1^{(4)}$ convolution and subsampling

- Thus, we use those as inputs for the fully connected layer of input.

Now assuming a single $k=1$ neuron

$$y_1^{(7)} = f\left(z_1^{(7)}\right)$$

$$z_1^{(7)} = \sum_{r=1}^{m_2^{(6)}} \sum_{s=1}^{m_3^{(6)}} w_{r,s}^{(7)} \left(Y_1^{(6)}\right)_{r,s}$$

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We have

That our final cost function is equal to

$$L(\boldsymbol{t}) = \frac{1}{2}\left(y_1^{(7)} - t_1^{(7)}\right)^2$$

Outline

After collecting all input/output

Therefore

- We have using sum of squared errors (loss function):

$$\min_{\boldsymbol{W}} H(\boldsymbol{W}) = \frac{1}{2}\left(y_1^{(7)} - t_1^{(7)}\right)^2$$

Therefore, we can obtain

$$\frac{\partial H(\boldsymbol{W})}{\partial w_{1,r,s}^{(7)}} = \frac{1}{2} \times \frac{\partial\left(y_1^{(7)} - t_1^{(7)}\right)^2}{\partial w_{1,r,s}^{(7)}}$$

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We get in the first part of the equation

$$\frac{\partial \left(t_1 - y_1^{(7)}\right)^2}{\partial w_{1,r,s}^{(7)}} = \left(y_1^{(7)} - t_1^{(7)}\right) \frac{\partial y_1^{(7)}}{\partial w_{1,r,s}^{(7)}}$$

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$$\frac{\partial f\left(z_1^{(7)}\right)}{\partial z_1^{(7)}} = \frac{e^{kz_1^{(7)}}}{\left(1+e^{kz_1^{(7)}}\right)}$$

Finally

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Now

Given the pooling

$$Y_1^{(6)} = Sf\left[\left(Y_1^{(4)}\right)\right]S^T$$

We have that

$$\left(Y_1^{(4)}\right)_{r,s} = \left(B_1^{(4)}\right)_{r,s} + \sum_{k=-h_1^{(l)}}^{h_1^{(l)}} \sum_{l=-h_2^{(l)}}^{h_2^{(l)}} \left(K_{11}^{(4)}\right)_{k,l} \left(Y^{(3)}\right)_{r+k,s+l}$$

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We have then

$$\frac{\partial H\left(\boldsymbol{W}\right)}{\partial\left(K_{11}^{(4)}\right)_{k,t}}=\frac{1}{2}\times\frac{\partial\left(y_{1}^{(7)}-t_{1}\right)^{2}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

We have the following chain of derivations

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$$\frac{\partial z_i^{(7)}}{\partial \left(Y_1^{(6)}\right)_{r,s}} = w_{r,s}^{(7)}$$

The final convolution is assuming that

$$\frac{\partial \left(Y_1^{(6)}\right)_{r,s}}{\partial \left(K_{11}^{(4)}\right)_{k,t}} = \frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial \left(K_{11}^{(4)}\right)_{k,t}}$$

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$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(K_{11}^{(4)}\right)_{k,t}} = \frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}} \times \frac{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

Then

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}} = f'\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]$$

Therefore

We have

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(K_{11}^{(4)}\right)_{k,t}}=\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}\times\frac{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

Then

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}=f'\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]$$

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Finally, we have

The equation

$$\frac{\partial \left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}{\partial \left(K_{11}^{(4)}\right)_{k,t}} = \left(Y^{(3)}\right)_{2(r-1)+k,2(s-1)+t}$$

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The Other Equations

I will leave you to devise them

- They are a repetitive procedure.

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