Introduction to Neural Networks and Deep Learning

Multilayer Perceptron

Andres Mendez-Vazquez

September 15, 2019

Outline

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Do you remember?

The Perceptron has the following problem

Given that the perceptron is a linear classifier

It is clear that

It will never be able to classify stuff that is not linearly separable

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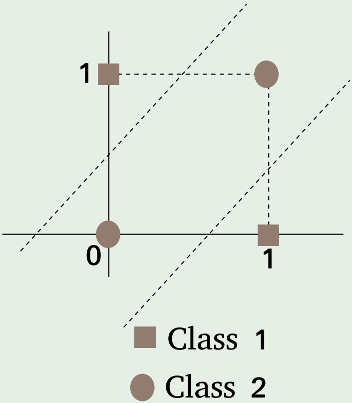
Example: XOR Problem

The Problem



The Perceptron cannot solve it

Because

The perceptron is a linear classifier!!!

Thus

Something needs to be done!!!

Maybe

Add an extra layer!!!

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A little bit of history

It was first cited by Vapnik

Vapnik cites (Bryson, A.E.; W.F. Denham; S.E. Dreyfus. Optimal programming problems with inequality constraints. I: Necessary conditions for extremal solutions. AIAA J. 1, 11 (1963) 2544-2550 [1]) as the first publication of the backpropagation algorithm in his book "Support Vector Machines."

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Arthur E. Bryson and Yu-Chi Ho described it as a multi-stage dynamic system optimization method in 1969.

However

It was not until 1974 and later, when applied in the context of neural networks and through the work of Paul Werbos, David E. Rumelhart, Geoffrey E. Hinton and Ronald J. Williams that it gained recognition [2, 3, 4].

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Then

Something Notable

It led to a “renaissance” in the field of artificial neural network research.

Nevertheless

During the 2000s it fell out of favor but has returned again in the 2010s, now able to train much larger networks using huge modern computing power such as GPUs.

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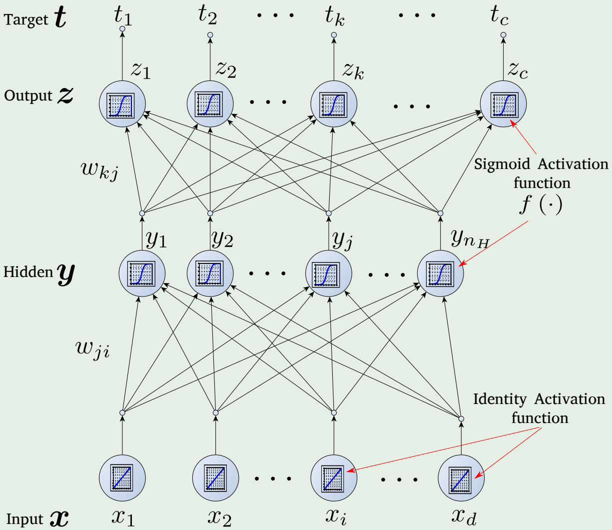
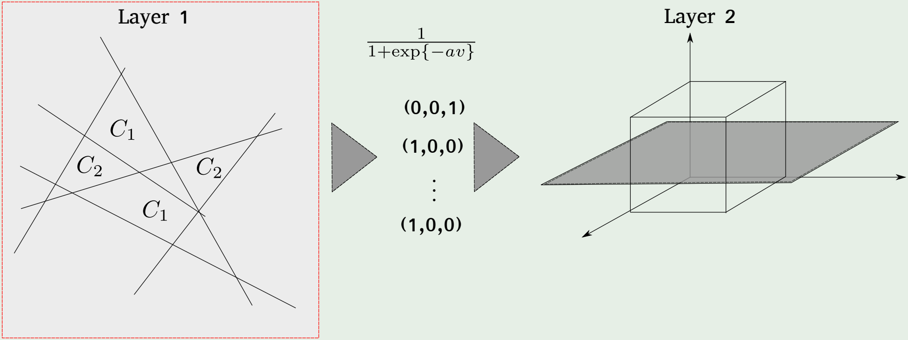
Multi-Layer Perceptron (MLP) [5]

Multi-Layer Architecture·



What do we believe happens in the MLP

We have something like this



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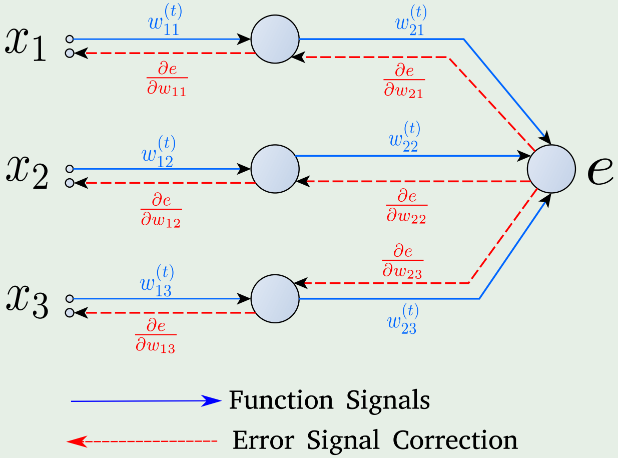
Information Flow

We have the following information flow



Based in the Previous Idea

People noticed that you require a way to get the info

- In order to build the error signal...

This can be done by simply

- Evaluating the function composition to get the error

$$x \longrightarrow f_n \circ f_{n-1} \circ \cdots f_1(x) = y \longrightarrow e = t - y$$

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Now, if we want to use the gradient descent

We have a small problem

- If you have the derivative of the cost function by weights that are deep into the network from the cost function

$$\frac{\partial F(\boldsymbol{x})}{\partial w} = \frac{\partial \left(t - f_n \circ f_{n-1} \circ \cdots f_1(\boldsymbol{x})\right)^2}{\partial w}$$

Therefore, we need to use the chain rule of derivatives

- To reach those functions to build the gradient descent for weights deep into the network

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Chain Rule

Definition

- Given a composition of differentiable functions

$$F\left(\boldsymbol{x}\right) = f_n \circ f_{n-1} \circ \cdots f_1\left(\boldsymbol{x}\right)$$

Then

$$\frac{\partial F\left(\boldsymbol{x}\right)}{\partial \boldsymbol{x}} = \frac{\partial f_n}{\partial f_{n-1}} \times \frac{\partial f_{n-1}}{\partial f_{n-2}} \times \frac{\partial f_{n-2}}{\partial f_{n-3}} \times \cdots \times \frac{\partial f_1}{\partial \boldsymbol{x}}$$

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Therefore

If we derivate with respect to weights at each level, we are actually back-propagating the error

$$J\left(\boldsymbol{W}\right) = \frac{1}{2}\left(t - z\right)^2$$

$$z = f\left(\boldsymbol{w}_1^T \boldsymbol{y}\right)$$

$$\boldsymbol{y} = g\left(\boldsymbol{w}_2^T \boldsymbol{x}\right)$$

Therefore, for the first layer

$$\frac{\partial J\left(\boldsymbol{W}\right)}{\partial w_1} = \frac{\partial J\left(\boldsymbol{W}\right)}{\partial z} \times \frac{\partial f\left(w_1^T y\right)}{\partial w_1^T y} \times \frac{\partial w_1^T y}{\partial w_1}$$

What about the second layer?

- We go to the blackboard!!!

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Outline

The Quadratic Learning Error Function

Cost Function our well know error at pattern m

$$J(m) = \frac{1}{2} e_k^2(m) \tag{1}$$

Delta Rule or Widrow-Hoff Rule

$$\Delta w_{kj}(m) = -\eta e_k(m) x_j(m) \tag{2}$$

Actually this is know as Gradient Descent

$$w_{kj}(m+1) = w_{kj}(m) + \Delta w_{kj}(m) \tag{3}$$

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Back-propagation

Setup

Let t_k be the k-th target (or desired) output and z_k be the k-th computed output with $k = 1, \ldots, d$ and $\boldsymbol{w}$ represents all the weights of the network

Training Error for a single Pattern or Sample!!!

$$J(\boldsymbol{w}) = \frac{1}{2}\sum_{k=1}^{c}(t_k - z_k)^2 = \frac{1}{2}\|\boldsymbol{t} - \boldsymbol{z}\|^2 \tag{4}$$

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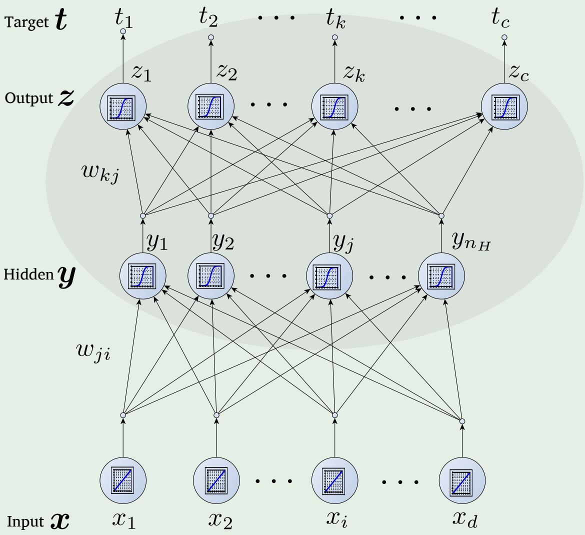
Multilayer Architecture

Multilayer Architecture: hidden–to-output weights



Observation about the activation function

Hidden Output is equal to

$$y_j = f\left(\sum_{i=1}^{d} w_{ji} x_i\right)$$

Output is equal to

$$z_k = f\left(\sum_{j=1}^{y_{n_H}} w_{kj} y_j\right)$$

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Hidden–to-Output Weights

net_k

- It describes how the overall error changes with the activation of the unit's net:

$$net_k = \sum_{j=1}^{y_{n_H}} w_{kj} y_j = \boldsymbol{w}_k^T \cdot \boldsymbol{y} \tag{5}$$

Which is composed with an activation function f

$$z_k = f\left(net_k\right) \tag{6}$$

Thus

$$\frac{\partial z_k}{\partial net_k} = f'\left(net_k\right) \tag{7}$$

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Now, we have the cost function J

Thus, we can apply the chain rule to the cost function

$$\frac{\partial J\left(\boldsymbol{z}_k\right)}{\partial w_{kj}}=\frac{\partial J\left(\boldsymbol{z}_k\right)}{\partial net_k}\cdot\frac{\partial net_k}{\partial w_{kj}}=-\delta_k\cdot\frac{\partial net_k}{\partial w_{kj}} \tag{8}$$

Still, we need to apply the same for $\frac{\partial J\left(\boldsymbol{z}_k\right)}{\partial net_k}$

$$\frac{\partial J\left(\boldsymbol{z}_k\right)}{\partial net_k}=\frac{\partial J\left(\boldsymbol{z}_k\right)}{\partial z_k}\cdot\frac{\partial z_k}{\partial net_k}=-\left(t_k-z_k\right)f'\left(net_k\right) \tag{9}$$

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Then

We create a new variable

$$\delta_k = -\frac{\partial J\left(\boldsymbol{z}_k\right)}{\partial net_k} = \left(t_k - z_k\right) f'\left(net_k\right)$$

Not only that, but we need

$$\frac{\partial net_k}{\partial w_{kj}}$$

Since $net_k = \boldsymbol{w}_k^T \cdot \boldsymbol{y}$ therefore:

$$\frac{\partial net_k}{\partial w_{kj}} = y_j \tag{10}$$

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Finally

The weight update (or learning rule) for the hidden-to-output weights is:

$$\triangle w_{kj} = \eta \delta_k y_j = \eta \left(t_k - z_k\right) f'\left(net_k\right) y_j \tag{11}$$

Outline

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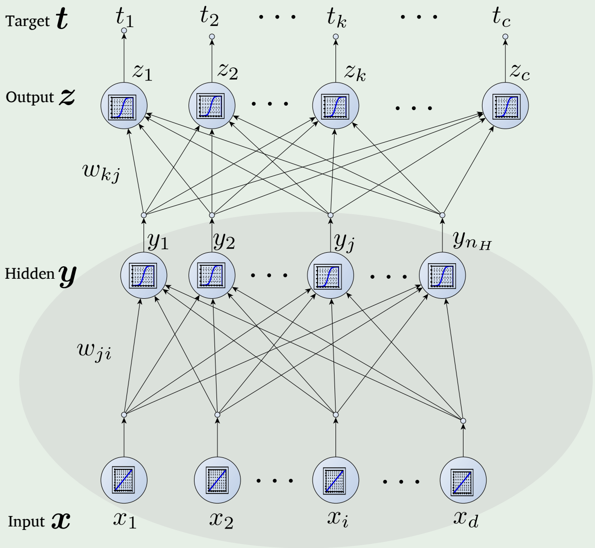
Multi-Layer Architecture

Going deeper into the network



Input–to-Hidden Weights

Chain rule on the Input–to-Hidden weights

$$\frac{\partial J}{\partial w_{ji}} = \frac{\partial J}{\partial y_j} \cdot \frac{\partial y_j}{\partial net_j} \cdot \frac{\partial net_j}{\partial w_{ji}} \tag{12}$$

Thus

$$\begin{aligned}
\frac{\partial J}{\partial y_j} &= \frac{\partial}{\partial y_j}\left[\frac{1}{2}\sum_{k=1}^{c}\left(t_k - z_k\right)^2\right] \\
&= -\sum_{k=1}^{c}\left(t_k - z_k\right)\frac{\partial z_k}{\partial y_j} \\
&= -\sum_{k=1}^{c}\left(t_k - z_k\right)\frac{\partial z_k}{\partial net_k} \cdot \frac{\partial net_k}{\partial y_j} \\
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&= -\sum_{k=1}^{c}\left(t_k - z_k\right)\frac{\partial f\left(net_k\right)}{\partial net_k} \cdot w_{kj}
\end{aligned}$$

Input–to-Hidden Weights

Finally

$$\frac{\partial J}{\partial y_j} = -\sum_{k=1}^{c} (t_k - z_k) f'(net_k) \cdot w_{kj} \tag{13}$$

Remember

$$\delta_k = -\frac{\partial J}{\partial net_k} = (t_k - z_k) f'(net_k) \tag{14}$$

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What is $\frac{\partial y_j}{\partial net_j}$?

First

$$net_j = \sum_{i=1}^{d} w_{ji} x_i = \boldsymbol{w}_j^T \cdot \boldsymbol{x} \tag{15}$$

Then

$$y_j = f(net_j)$$

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Then, we can define δ_j

By defying the sensitivity for a hidden unit:

$$\delta_j = f'(net_j)\sum_{k=1}^{c} w_{kj}\delta_k \tag{16}$$

Which means that:

"The sensitivity at a hidden unit is simply the sum of the individual sensitivities at the output units weighted by the **hidden-to-output weights** w_{kj}; all multiplied by $f'(net_j)$"

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What about $\frac{\partial net_j}{\partial w_{ji}}$?

We have that

$$\frac{\partial net_j}{\partial w_{ji}} = \frac{\partial \boldsymbol{w}_j^T \cdot \boldsymbol{x}}{\partial w_{ji}} = \frac{\partial \sum_{i=1}^{d} w_{ji} x_i}{\partial w_{ji}} = x_i$$

Finally

The learning rule for the input-to-hidden weights is:

$$\Delta w_{ji} = \eta x_i \delta_j = \eta \left[\sum_{k=1}^{c} w_{kj} \delta_k \right] f' \left(net_j \right) x_i \tag{17}$$

Basically, the entire training process has the following steps

Initialization

Assuming that no prior information is available, pick the synaptic weights and thresholds

Forward Computation

Compute the induced function signals of the network by proceeding forward through the network, layer by layer.

Backward Computation

Compute the local gradients of the network.

Finally

Adjust the weights!!!

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Problems with Hidden Layers

1. Increase complexity of Training
2. It is necessary to think about "Long and Narrow" network vs "Short and Fat" network.

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Intuition for a One Hidden Layer

1. For every input case of region, that region can be delimited by hyperplanes on all sides using hidden units on the first hidden layer.
2. A hidden unit in the second layer than ANDs them together to bound the region.

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Now, Calculating Total Change

We have for that

The Total Training Error is the sum over the errors of N individual patterns

The Total Training Error

$$J = \sum_{p=1}^{N} J_p = \frac{1}{2} \sum_{p=1}^{N} \sum_{k=1}^{d} \left(t_k^p - z_k^p\right)^2 = \frac{1}{2} \sum_{p=1}^{n} \left\| t^p - z^p \right\|^2 \qquad (18)$$

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About the Total Training Error

Remarks

- A weight update may reduce the error on the single pattern being presented but can increase the error on the full training set.
- However, given a large number of such individual updates, the total error of equation (18) decreases.

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Now, we want the training to stop

Therefore

It is necessary to have a way to stop when the change of the weights is enough!!!

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A simple way to stop the training

- The algorithm terminates when the change in the criterion function $J(\boldsymbol{w})$ is smaller than some preset value Θ.

$$\Delta J(\boldsymbol{w}) = |J(\boldsymbol{w}(t+1)) - J(\boldsymbol{w}(t))| \tag{19}$$

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It is necessary to have a way to stop when the change of the weights is enough!!!

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Other Stopping Criteria

Norm of the Gradient

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About the Stopping Criteria

Observations

1. Before training starts, the error on the training set is high.
    - Through the learning process, the error becomes smaller.
2. The error per pattern depends on the amount of training data and the expressive power (such as the number of weights) in the network.
3. The average error on an independent test set is always higher than on the training set, and it can decrease as well as increase.
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Epoch

As with other types of backpropagation, 'learning' is a supervised process that occurs with each cycle or 'epoch' through a forward activation flow of outputs, and the backwards error propagation of weight adjustments.

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Outline

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Final Basic Batch Algorithm

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1. **Initialize** random w, number of hidden units n_H, number of outputs z, stopping criterion Θ, learning rateη, epoch $m = 0$
2. **do**
3. $\quad m = m + 1$
4. $\quad\quad$ **for** $s = 1$ **to** N
5. $\quad\quad\quad \boldsymbol{x}(m) = \boldsymbol{X}(:, s)$
6. $\quad\quad\quad$ **for** $k = 1$ **to** c
7. $\quad\quad\quad\quad \delta_k = (t_k - z_k)\, f'\left(\boldsymbol{w}_k^T \cdot \boldsymbol{y}\right)$
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Final Basic Batch Algorithm

Perceptron$(\boldsymbol{X})$

1. **Initialize random** w, **number of hidden units** n_H, **number of outputs** $\boldsymbol{z}$, **stopping criterion** Θ, **learning rate**η, **epoch**

   $m = 0$
2. **do**
3. $\quad m = m + 1$
4. $\quad\quad$ **for** $s = 1$ **to** N
5. $\quad\quad\quad \boldsymbol{x}(m) = \boldsymbol{X}(:, s)$
6. $\quad\quad\quad$ **for** $k = 1$ **to** c
7. $\quad\quad\quad\quad \delta_k = (t_k - z_k) f'\left(\boldsymbol{w}_k^T \cdot \boldsymbol{y}\right)$
8. $\quad\quad\quad\quad$ **for** $j = 1$ **to** n_H
9. $\quad\quad\quad\quad\quad net_j = \boldsymbol{w}_j^T \cdot \boldsymbol{x}; y_j = f\left(net_j\right)$
10. $\quad\quad\quad\quad\quad w_{kj}(m) = w_{kj}(m) + \eta \delta_k y_j(m)$
11. $\quad\quad\quad$ **for** $j = 1$ **to** n_H
12. $\quad\quad\quad\quad \delta_j = f'\left(net_j\right) \sum_{k=1}^{c} w_{kj} \delta_k$
13. $\quad\quad\quad\quad$ **for** $i = 1$ **to** d
14. $\quad\quad\quad\quad\quad w_{ji}(m) = w_{ji}(m) + \eta \delta_j x_i(m)$
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Outline

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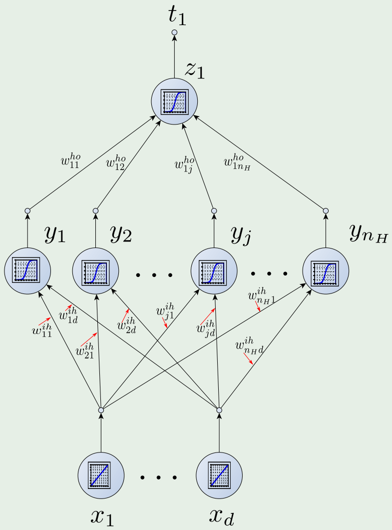
Example of Architecture to be used

Given the following Architecture and assuming N samples



Outline

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Generating the output z_k

Given the input

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_N \end{bmatrix} \tag{22}$$

Where

x_i is a vector of features

$$x_i = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{di} \end{pmatrix} \tag{23}$$

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Therefore

We must have the following matrix for the input to hidden inputs

$$
\boldsymbol{W}_{IH} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1d} \\ w_{21} & w_{22} & \cdots & w_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n_H 1} & w_{n_H 2} & \cdots & w_{n_H d} \end{pmatrix} = \begin{pmatrix} \boldsymbol{w}_1^T \\ \boldsymbol{w}_2^T \\ \vdots \\ \boldsymbol{w}_{n_H}^T \end{pmatrix} \tag{24}
$$

Given that $\boldsymbol{w}_j = \begin{pmatrix} w_{j1} \\ w_{j2} \\ \vdots \\ w_{jd} \end{pmatrix}$

Thus

We can create the net_j for all the inputs by simply

$$
\boldsymbol{net}_j = \boldsymbol{W}_{IH}\boldsymbol{X} = \begin{pmatrix} \boldsymbol{w}_1^T\boldsymbol{x}_1 & \boldsymbol{w}_1^T\boldsymbol{x}_2 & \cdots & \boldsymbol{w}_1^T\boldsymbol{x}_N \\ \boldsymbol{w}_2^T\boldsymbol{x}_1 & \boldsymbol{w}_2^T\boldsymbol{x}_2 & \cdots & \boldsymbol{w}_2^T\boldsymbol{x}_N \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{w}_{n_H}^T\boldsymbol{x}_1 & \boldsymbol{w}_{n_H}^T\boldsymbol{x}_2 & \cdots & \boldsymbol{w}_{n_H}^T\boldsymbol{x}_N \end{pmatrix} \tag{25}
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We can create the $\boldsymbol{net}_j$ for all the inputs by simply

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Now, we need to generate the $\boldsymbol{y}_k$

We apply the activation function element by element in $\boldsymbol{net}_j$

$$\boldsymbol{y}_1 = \begin{pmatrix} f\left(\boldsymbol{w}_1^T\boldsymbol{x}_1\right) & f\left(\boldsymbol{w}_1^T\boldsymbol{x}_2\right) & \cdots & f\left(\boldsymbol{w}_1^T\boldsymbol{x}_N\right) \\ f\left(\boldsymbol{w}_2^T\boldsymbol{x}_1\right) & f\left(\boldsymbol{w}_2^T\boldsymbol{x}_2\right) & \cdots & f\left(\boldsymbol{w}_2^T\boldsymbol{x}_N\right) \\ \vdots & \vdots & \ddots & \vdots \\ f\left(\boldsymbol{w}_{n_H}^T\boldsymbol{x}_1\right) & f\left(\boldsymbol{w}_{n_H}^T\boldsymbol{x}_2\right) & \cdots & f\left(\boldsymbol{w}_{n_H}^T\boldsymbol{x}_N\right) \end{pmatrix} \tag{26}$$

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IMPORTANT about overflows!!!

- Be careful about the numeric stability of the activation function.
- I the case of python, we can use the ones provided by scipy.special

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However, We can create a Sigmoid function

It is possible to use the following pseudo-code

Sigmoid(x)

1. if try $\left\{\frac{1.0}{1.0+\exp\{-\alpha x\}}\right\}$ catch $\{OVERFLOW\}$ ◁ I will use a
   ◁ try-and-catch to catch
   ◁ the overflow
2. if $x < 0$
3. return 0
4. else
5. return 1
6. else
7. return $\frac{1.0}{1.0+\exp\{-\alpha x\}}$ ◁ 1.0 refers to the floating point (Rationals
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For this, we get $\boldsymbol{net}_k$

For this, we obtain the $\boldsymbol{W}_{HO}$

$$\boldsymbol{W}_{HO} = \begin{pmatrix} w^o_{11} & w^o_{12} & \cdots & w^o_{1n_H} \end{pmatrix} = \begin{pmatrix} \boldsymbol{w}^T_o \end{pmatrix} \tag{27}$$

Thus

$$\boldsymbol{net}_k = \begin{pmatrix} w^o_{11} & w^o_{12} & \cdots & w^o_{1n_H} \end{pmatrix} \begin{pmatrix} f\left(\boldsymbol{w}_1^T \boldsymbol{x}_1\right) & f\left(\boldsymbol{w}_1^T \boldsymbol{x}_2\right) & \cdots & f\left(\boldsymbol{w}_1^T \boldsymbol{x}_N\right) \\ f\left(\boldsymbol{w}_2^T \boldsymbol{x}_1\right) & f\left(\boldsymbol{w}_2^T \boldsymbol{x}_2\right) & \cdots & f\left(\boldsymbol{w}_2^T \boldsymbol{x}_N\right) \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{f\left(\boldsymbol{w}_{n_H}^T \boldsymbol{x}_1\right)}_{\boldsymbol{y}_{k1}} & \underbrace{f\left(\boldsymbol{w}_{n_H}^T \boldsymbol{x}_2\right)}_{\boldsymbol{y}_{k2}} & \cdots & \underbrace{f\left(\boldsymbol{w}_{n_H}^T \boldsymbol{x}_N\right)}_{\boldsymbol{y}_{kN}} \end{pmatrix} \tag{28}$$

In matrix notation

$$\boldsymbol{net}_k = \begin{pmatrix} \boldsymbol{w}_o^T \boldsymbol{y}_{k1} & \boldsymbol{w}_o^T \boldsymbol{y}_{k2} & \cdots & \boldsymbol{w}_o^T \boldsymbol{y}_{kN} \end{pmatrix} \tag{29}$$

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Outline

Now, we have

Thus, we have $\boldsymbol{z}_k$ (In our case $k = 1$, but it could be a range of values)

$$\boldsymbol{z}_k = \begin{pmatrix} f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) & f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k2}\right) & \cdots & f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{kN}\right) \end{pmatrix} \tag{30}$$

Thus, we generate a vector of differences

$$\boldsymbol{d} = \boldsymbol{t} - \boldsymbol{z}_k = \begin{pmatrix} t_1 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) & t_2 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k2}\right) & \cdots & t_N - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{kN}\right) \end{pmatrix} \tag{31}$$

where $\boldsymbol{t} = \begin{pmatrix} t_1 & t_2 & \cdots & t_N \end{pmatrix}$ is a row vector of desired outputs for each sample.

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Now, we multiply element wise

We have the following vector of derivatives of net

$$\boldsymbol{D}_f = \begin{pmatrix} \eta f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) & \eta f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k2}\right) & \cdots & \eta f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{kN}\right) \end{pmatrix} \quad (32)$$

where η is the step rate.

Finally, by element wise multiplication (Hadamard Product)

$$\boldsymbol{d} = \left(\eta\left[t_1 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) \quad \eta\left[t_2 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k2}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k2}\right) \quad \cdots \quad \eta\left[t_N - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{kN}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{kN}\right) \right)$$

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Tile $\boldsymbol{d}$

Tile downward

$$\boldsymbol{d}_{tile} = n_H \text{ rows} \left\{ \begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{d} \\ \vdots \\ \boldsymbol{d} \end{pmatrix} \right. \tag{33}$$

Finally, we multiply element wise against $\boldsymbol{y}_1$ (Hadamard Product)

$$\Delta w_{1j}^{temp} = \boldsymbol{y}_1 \circ \boldsymbol{d}_{tile} \tag{34}$$

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$$\Delta \boldsymbol{w}_{1j}^{temp} = \boldsymbol{y}_1 \circ \boldsymbol{d}_{tile} \tag{34}$$

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We obtain the total $\Delta \boldsymbol{w}_{1j}$

We sum along the rows of $\Delta \boldsymbol{w}_{1j}^{temp}$

$$\Delta \boldsymbol{w}_{1j} = \begin{pmatrix} \eta \left[t_1 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) y_{11} + \eta \left[t_1 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) y_{1N} \\ \vdots \\ \eta \left[t_1 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) y_{n_H 1} + \eta \left[t_1 - f\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right)\right] f'\left(\boldsymbol{w}_o^T \boldsymbol{y}_{k1}\right) y_{n_H N} \end{pmatrix} \quad (35)$$

where $y_{hm} = f\left(\boldsymbol{w}_h^T \boldsymbol{x}_m\right)$ with $h = 1, 2, ..., n_H$ and $m = 1, 2, ..., N$.

Finally, we update the first weights

We have then

$$\boldsymbol{W}_{HO}(t+1) = \boldsymbol{W}_{HO}(t) + \Delta \boldsymbol{w}_{1j}^{T}(t) \quad (36)$$

Outline

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First

We multiply element wise the $\boldsymbol{W}_{HO}$ and $\Delta \boldsymbol{w}_{1j}$

$$\boldsymbol{T} = \Delta \boldsymbol{w}_{1j}^{T} \circ \boldsymbol{W}_{HO}^{T} \tag{37}$$

Now, we obtain the element wise derivative of net_j

$$\boldsymbol{Dnet}_j = \begin{pmatrix} f'\left(\boldsymbol{w}_1^T \boldsymbol{x}_1\right) & f'\left(\boldsymbol{w}_1^T \boldsymbol{x}_2\right) & \cdots & f'\left(\boldsymbol{w}_1^T \boldsymbol{x}_N\right) \\ f'\left(\boldsymbol{w}_2^T \boldsymbol{x}_1\right) & f'\left(\boldsymbol{w}_2^T \boldsymbol{x}_2\right) & \cdots & f'\left(\boldsymbol{w}_2^T \boldsymbol{x}_N\right) \\ \vdots & \vdots & \ddots & \vdots \\ f'\left(\boldsymbol{w}_{n_H}^T \boldsymbol{x}_1\right) & f'\left(\boldsymbol{w}_{n_H}^T \boldsymbol{x}_2\right) & \cdots & f'\left(\boldsymbol{w}_{n_H}^T \boldsymbol{x}_N\right) \end{pmatrix} \tag{38}$$

First

We multiply element wise the $\boldsymbol{W}_{HO}$ and $\Delta \boldsymbol{w}_{1j}$

$$\boldsymbol{T} = \Delta \boldsymbol{w}_{1j}^{T} \circ \boldsymbol{W}_{HO}^{T} \tag{37}$$

Now, we obtain the element wise derivative of $\boldsymbol{net}_j$

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Thus

We tile to the right T

$$\boldsymbol{T}_{tile} = \underbrace{\begin{pmatrix} \boldsymbol{T} & \boldsymbol{T} & \cdots & \boldsymbol{T} \end{pmatrix}}_{N \text{ Columns}} \tag{39}$$

Now, we multiply element wise together with η

$$\boldsymbol{P}_t = \eta \left(\boldsymbol{Dnet}_j \circ \boldsymbol{T}_{tile} \right) \tag{40}$$

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Finally

We get use the transpose of $\boldsymbol{X}$ which is a $N \times d$ matrix

$$\boldsymbol{X}^T = \begin{pmatrix} \boldsymbol{x}_1^T \\ \boldsymbol{x}_2^T \\ \vdots \\ \boldsymbol{x}_N^T \end{pmatrix} \quad (41)$$

Finally, we get a $n_H \times d$ matrix

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Thus, given $\boldsymbol{W}_{IH}$

$$\boldsymbol{W}_{IH}(t+1) = \boldsymbol{W}_{HO}(t) + \Delta w_{ij}^T(t) \quad (43)$$

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Outline

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Two ways of achieving this, LeCun 1993

- The use of an example that results in the largest training error.
- The use of an example that is radically different from all those previously used.

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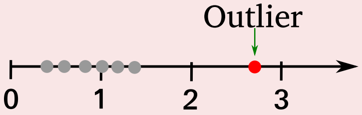

Outline

Cinvestav

Activation Functions [5]

We say that

An activation function $f(v)$ is antisymmetric if $f(-v) = -f(v)$

It seems to be

That the multilayer perceptron learns faster using an antisymmetric function.

- For example, the hyperbolic tangent function

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Hyperbolic tangent function

Another commonly used form of sigmoid non linearity is the hyperbolic tangent function

$$f_j\left(v_j\left(n\right)\right) = a\tanh\left(bv_j\left(n\right)\right) \tag{44}$$

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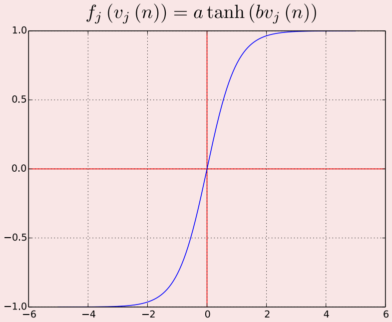

The differential of the hyperbolic tangent

We have

$$\begin{aligned} f_j\left(v_j\left(n\right)\right) =& ab\,\mathsf{sech}^2\left(bv_j\left(n\right)\right) \\ =& ab\left(1-\tanh^2\left(bv_j\left(n\right)\right)\right) \end{aligned}$$

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Logistic Function

This non-linear function has the following definition for a neuron j

$$f_j\left(v_j\left(n\right)\right) = \frac{1}{1+\exp\left\{-av_j\left(n\right)\right\}} \; a > 0 \text{ and } -\infty < v_j\left(n\right) < \infty \tag{45}$$

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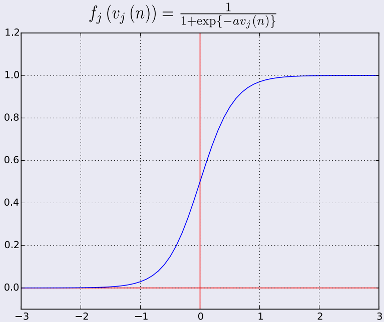

The differential of the sigmoid function

Now, if we differentiate it, we have

$$f_j'\left(v_j\left(n\right)\right)=\left[\frac{1}{1+\exp\left\{-av_j\left(n\right)\right\}}\right]\left[1-\frac{1}{1+\exp\left\{-av_j\left(n\right)\right\}}\right]$$

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The outputs finish as

For the output neurons

$$\delta_k = (t_k - z_k) f'(net_k)$$

$$= (t_k - f_k(v_k(n))) f_k(v_k(n)) (1 - f_k(v_k(n)))$$

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Problems

It is clear from our first class

- The problems of the sigmoid activation function, the vanishing gradient

Not only that, we want activation functions

- That accelerate the learning
- That can control the previous problem
- An any other possible properties

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Something Notable

- This activation function was first introduced to a dynamical network by Hahnloser et al. [6].

Finally eleven years after

- It was proved that the ReLU function could accelerate the training of deep networks [7].

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Rectified Linear Unit (ReLU)

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ReLU &: \mathbb{R} \longrightarrow \mathbb{R} \\
ReLU\left(x\right) &= \max\left(0, x\right), \; x \in \mathbb{R}
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Plotting ReLU

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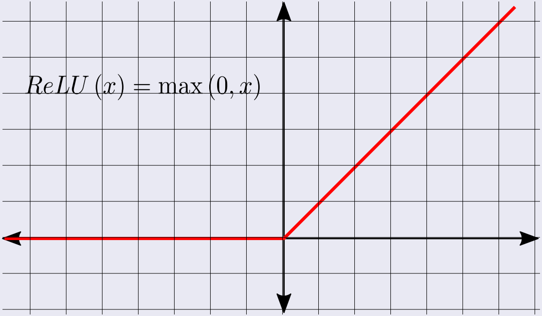
Plotting ReLU



Noisy ReLUs

Definition

$$f(x) = \max(0, x + Y) \text{ with } Y \sim N(0, \sigma(x))$$

They have been used

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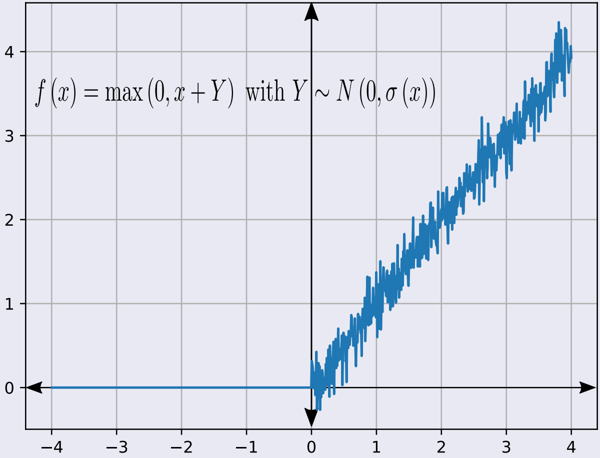

Leaky ReLU

Definition

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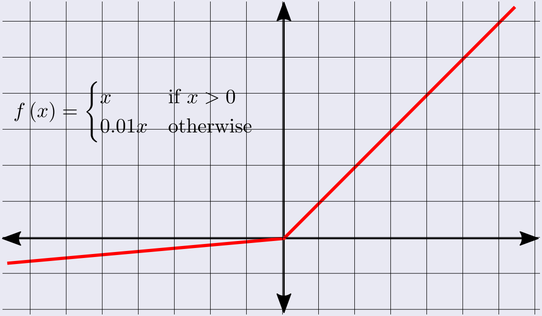

Furthermore

The Parametric ReLU

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ ax & \text{otherwise} \end{cases}$$

- with $a \leq 1$

ELU

Definition

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ a\left(e^{x} - 1\right) & \text{otherwise} \end{cases}$$

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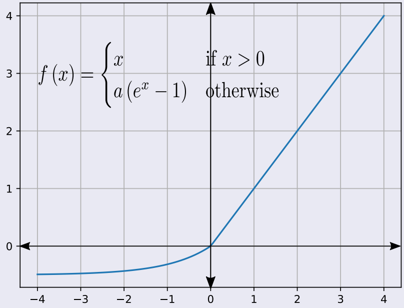

Outline

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Target Values

Important

It is important that the target values be chosen within the range of the sigmoid activation function.

Specifically

The desired response for neuron in the output layer of the multilayer perceptron should be offset by some amount ϵ

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For example

Given the a limiting value

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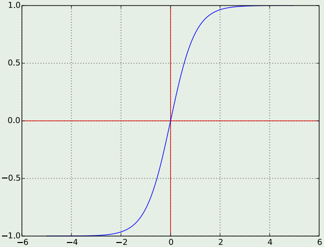

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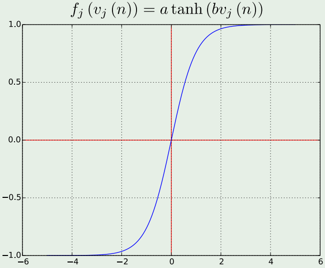

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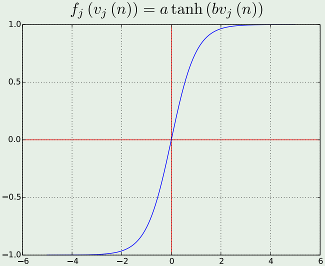

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Each input variable should be preprocessed so that:

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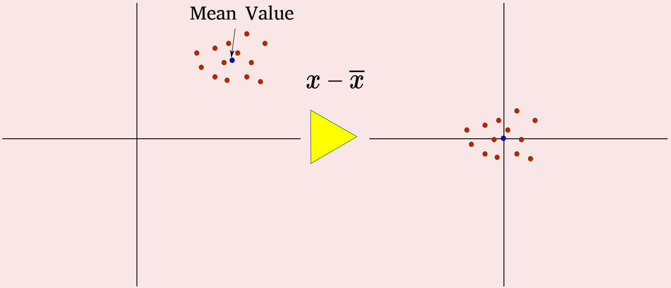

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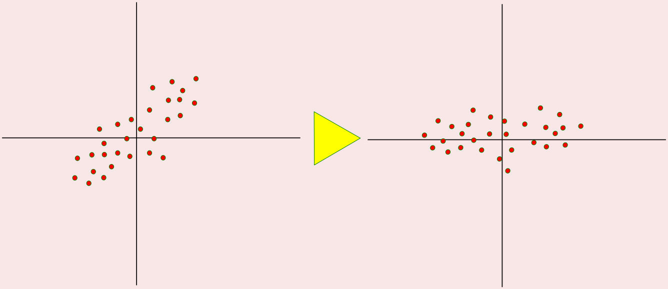

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- Learning form hints
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Why this is advocated in Artificial Neural Networks

First

Artificial neural networks that perform local computations are often held up as metaphors for biological neural networks.

Second

The use of local computations permits a graceful degradation in performance due to hardware errors, and therefore provides the basis for a fault-tolerant network design.

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- Back-propagation learning implies the existence of a "teacher," which in the con text of the brain would presumably be another set of neurons with novel properties.
- The existence of such neurons is biologically implausible.

However, all this has been seriously questioned [8, 9, 10]

Fifth

- Back-propagation learning implies the existence of a "teacher," which in the con text of the brain would presumably be another set of neurons with novel properties.
- The existence of such neurons is biologically implausible.

Computational Efficiency

Something Notable

The computational complexity of an algorithm is usually measured in terms of the number of multiplications, additions, and storage involved in its implementation.

- This is the electrical engineering approach!!!

Taking in account the total number of synapses, W including biases

We have $\triangle w_{kj} = \eta \delta_k y_j = \eta \left(t_k - z_k\right) f'\left(net_k\right) y_j$ (Backward Pass)

We have that for this step

1. We need to calculate net_k linear in the number of weights.
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Computational Efficiency

Now the Forward Pass

$$\Delta w_{ji} = \eta x_i \delta_j = \eta f'\left(net_j\right)\left[\sum_{k=1}^{c} w_{kj}\delta_k\right] x_i$$

We have that for this step

$\left[\sum_{k=1}^{c} w_{kj}\delta_k\right]$ takes, because of the previous calculations of δ_k's, linear on the number of weights

Clearly all this takes to have memory

In addition the calculation of the derivatives of the activation functions, but assuming a constant time.

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We have that

The Complexity of the multi-layer perceptron is

$$O(W) \text{ Complexity}$$

Exercises

We have from NN by Haykin

4.2, 4.3, 4.6, 4.8, 4.16, 4.17, 3.7

Outline

Introduction

Representation of functions

The main result in multi-layer perceptron is its power of representation.

Furthermore

After all, it is quite striking if we can represent continuous functions of the form $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ as a finite sum of simple functions.

Introduction

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After all, it is quite striking if we can represent continuous functions of the form $f : \mathbb{R}^n \longmapsto \mathbb{R}$ as a finite sum of simple functions.

Therefore

Our main goal

We want to know under which conditions the sum of the form:

$$G(\boldsymbol{x}) = \sum_{j=1}^{N} \alpha_j f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta_j\right) \tag{46}$$

can represent continuous functions in a specific domain.

Setup of the problem

Definition of I_n

It is an n-dimensional unit cube $[0,1]^n$

In addition, we have the following set of functions

$$C(I_n) = \{f : I_n \to \mathbb{R} | f \text{ is a continous function}\} \quad (47)$$

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Outline

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Now, some important definitions

Definition (Topological Space)

A topological space is then a set X together with a collection of subsets of X, called **open sets** and satisfying the following axioms:

1. The empty set and X itself are open.
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Then

We are interested in defining open and close sets in metric spaces

- After all this will allow to define the concept of closed set

Definition

- A subset U of a metric space (M, d) is called open if, given any point $x \in U$, there exists a real number $\epsilon > 0$ such that, given any point $y \in M$ with $d(x, y) < \epsilon$, y also belongs to U.

Therefore

- A set $V \subset M$ is closed if $M - V$ is open.

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Theorem

A compact set is closed and bounded.

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I_n is a compact set in $\mathbb{R}^n$.

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A function $f : X \to Y$ where the pre-image of every open set in Y is open in X.

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We have the following statement

Let K be a nonempty subset of $\mathbb{R}^n$, where $n > 1$. Then If K is compact, then every continuous real-valued function defined on K is **bounded.**

Definition (Supremum Norm)

Let X be a topological space and let F be the space of all bounded complex-valued continuous functions defined on K. The supremum norm is the norm defined on F by

$$\|f\| = \sup_{x \in X} |f(x)| \quad (48)$$

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Limit Points

Definition

If X is a topological space and p is a point in X, a neighborhood of p is a subset V of X that includes an open set U containing p, $p \in U \subseteq V$.

- This is also equivalent to $p \in X$ being in the interior of V.

Example in a metric space

In a metric space (X, d), a set V is a neighborhood of a point p if there exists an open ball with center at p and radius $r > 0$, such that

$$B_r(p) = B(p; r) = \{x \in X | d(x, p) < r\} \quad (49)$$

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Limit Points

Definition of a Limit Point

Let S be a subset of a topological space X. A point $x \in X$ is a limit point of S if every neighborhood of x contains at least one point of S different from x itself.

Example in $\mathbb{R}$

Which are the limit points of the set $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$?

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Which are the limit points of the set $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$?

This allows to define the idea of density

Something Notable

A subset A of a topological space X is dense in X, if for any point $x \in X$, any neighborhood of x contains at least one point from A.

Classic Example

The real numbers with the usual topology have the rational numbers as a countable dense subset.

- Why do you believe the floating-point numbers are rational?

In addition

Also the irrational numbers.

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Definition

The closure of a set S is the set of all points of closure of S, that is, the set S together with all of its limit points.

Example

The closure of the following set $(0,1) \cup \{2\}$

Meaning

Not all points in the closure are limit points.

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Hausdorff Space

Definition of Separation

Points x and y in a topological space X can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint.

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Now, the Measure Concept

Definition of $\sigma-$algebra

Let $\mathcal{A} \subset \mathcal{P}(X)$, we say that $\mathcal{A}$ to be an algebra if

1. $\emptyset,\ X \in \mathcal{A}$.
2. $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.
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In $X = [0, 1)$, the class $\mathcal{A}_0$ consisting of $\emptyset$, and all finite unions $A = \cup_{i=1}^{n} [a_i, b_i)$ with $0 \leq a_i < b_i \leq a_{i+1} \leq 1$ is an algebra.

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Definition of additivity

Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(\emptyset) = 0$, we say that μ is $\sigma-$additive if for any $\{A_i\}_{i \in I} \subset \mathcal{A}$ (Where I can be finite of infinite countable) of mutually disjoint sets such that $\cup_{i \in I} A_i \in \mathcal{A}$, we have that

$$\mu\left(\cup_{i \in I} A_i\right) = \sum_{i \in I} \mu\left(A_i\right) \tag{50}$$

Definition of Measurability

Let $\mathcal{A}$ be a $\sigma-$algebra of subsets of X, we say that the [air $(X, \mathcal{A})$ is a measurable space where a $\sigma-$additive function $\mu : \mathcal{A} \to [0, +\infty]$ is called a measure on $(X, \mathcal{A})$.

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Definition

The Borel σ-algebra is defined to be the σ-algebra generated by the open sets (or equivalently, by the closed sets).

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Definition of a Borel Measure

If $\mathcal{F}$ is the Borel σ-algebra on some topological space, then a measure $\mu : \mathcal{F} \to \mathbb{R}$ is said to be a Borel measure (or Borel probability measure). For a Borel measure, all continuous functions are measurable.

1. $\mu(\emptyset) = 0$
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Regularity

A measure μ is Borel regular measure:

1. For every Borel set $B \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$, $\mu(A) = \mu(A \cap B) + \mu(A - B)$.
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Outline

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Discriminatory Functions

Definition

Given the set $M(I_n)$ of signed regular Borel measures, a function f is discriminatory if for a measure $\mu \in M(I_n)$

$$\int_{I_n} f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta\right) d\mu = 0 \tag{51}$$

for all $\boldsymbol{w} \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$ implies that $\mu = 0$

Definition

We say that f is sigmoidal if

$$f(t) \to \begin{cases} 1 & \text{as } t \to +\infty \\ 0 & \text{as } t \to -\infty \end{cases}$$

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The Important Theorem

Theorem 1

Let f a be any continuous discriminatory function. Then finite sums of the form

$$G(\boldsymbol{x}) = \sum_{j=1}^{N} \alpha_j f\left(\boldsymbol{w}_j^T \boldsymbol{x} + \theta_j\right), \tag{52}$$

where $\boldsymbol{w}_j \in \mathbb{R}^n$ and α_j, $\theta_j \in \mathbb{R}$ are fixed, are dense in $C(I_n)$

Meaning

Basically given any function $g \in C(I_n)$ and any neighborhood V of g, you have a $G \in V$.

The Important Theorem

Theorem 1

Let f a be any continuous discriminatory function. Then finite sums of the form

$$G\left(\boldsymbol{x}\right)=\sum_{j=1}^{N}\alpha_{j}f\left(\boldsymbol{w}_{j}^{T}\boldsymbol{x}+\theta_{j}\right), \tag{52}$$

where $\boldsymbol{w}_j \in \mathbb{R}^n$ and α_j, $\theta_j \in \mathbb{R}$ are fixed, are dense in $C\left(I_n\right)$

Meaning

Basically given any function $g \in C\left(I_n\right)$ and any neighborhood V of g, you have a $G \in V$.

Furthermore

In other words

Given any $g \in C(I_n)$ and $\epsilon > 0$, there is a sum, $G(\boldsymbol{x})$, of the above form, for which

$$|G(\boldsymbol{x}) - g(\boldsymbol{x})| < \epsilon \; \forall \boldsymbol{x} \in I_n \tag{53}$$

Proof

Let $S \subset C(I_n)$ be the set of functions of the form $G(\boldsymbol{x})$

First, S is a linear subspace of $C(I_n)$

Definition

A subset V of $\mathbb{R}^n$ is called a linear subspace of $\mathbb{R}^n$ if V contains the zero vector, and is closed under vector addition and scaling. That is, for $X, Y \in V$ and $c \in \mathbb{R}$, we have $X + Y \in V$ and $cX \in V$.

We claim that the closure of S is all of $C(I_n)$

Assume that the closure of S is not all of $C(I_n)$

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- If $p : V \to \mathbb{R}$ is a sublinear function
  - $p(x+y) \leq p(x) + p(y)$
  - $p(\alpha x) = \alpha p(x)$

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And $\varphi : U \to \mathbb{R}$ is a linear functional on a linear subspace $U \subseteq V$

- It is dominated by p on U, i.e. $\varphi(x) \leq p(x)$ $\forall x \in U$.

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Hahn-Banach Theorem

Then

There exists a **linear extension** $\psi : V \to \mathbb{R}$ of φ to the whole space V, i.e., there exists a linear functional ψ such that

1. $\psi(x) = \varphi(x)\ \forall x \in U$.
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Proof

It is possible to construct sublinear function defined as follow

We define the following linear functional

$$T(f) = \begin{cases} f & \text{if } f \in C(I_n) - R \\ 0 & \text{if } f \in R \end{cases} \tag{54}$$

Then

We have there is a bounded linear functional (Using T as p and φ, and $V \in C(I_n)$ and $U = R$) called $L \neq 0$ with $L(R) = L(S) = 0$.

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Proof

Now, we use the Riesz Representation Theorem

Let X be a locally compact Hausdorff space. For any positive linear functional ψ on $C(X)$, there is a unique regular Borel measure μ on X such that

$$\psi = \int_X f(x)\, d\mu(x) \tag{55}$$

for all f in $C(X)$

We can then do the following

$$L(h) = \int_{I_n} h(x)\, d\mu(x) \tag{56}$$

Where?

For some $\mu \in M(I_n)$, for all $h \in C(I_n)$

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Proof

In particular

Given that $f\left(\boldsymbol{w}^T\boldsymbol{x}+\theta\right)$ is in R for all $\boldsymbol{w}$ and θ

We must have that

$$\int_{I_n} f\left(\boldsymbol{w}^T\boldsymbol{x}+\theta\right) d\mu\left(\boldsymbol{x}\right)=0 \tag{57}$$

for all $\boldsymbol{w}$ and θ

But we assumed that f is discriminatory!!!

Then... $\mu=0$ contradicting the fact that $L\neq 0$!!! We have a contradiction!!!

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Proof

Finally

The subspace S of sums of the form G is dense!!!

Now, we deal with the sigmoidal function

Lemma 1

Any bounded, measurable sigmoidal function, f, is discriminatory. In particular, any continuous sigmoidal function is discriminatory.

Proof

I will leave this to you... it is possible I will get a question from this proof for the firs midterm.

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Outline

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We have the theorem finally!!!

Universal Representation Theorem for the multi-layer perceptron

Let f be any continuous sigmoid function. Then finite sums of the form

$$G(\boldsymbol{x}) = \sum_{j=1}^{N} \alpha_j f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta_j\right) \tag{58}$$

are dense in $C(I_n)$.

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Given any $g \in C(I_n)$ and $\epsilon > 0$, there is a sum $G(x)$ of the above form, for which

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Poof

Simple

Combine the theorem and lemma 1... and because the continuous sigmoid satisfy the conditions of the lemma... we have our representation!!!

A. E. Bryson, W. F. Denham, and S. E. Dreyfus, "Optimal programming problems with inequality constraints," *AIAA journal*, vol. 1, no. 11, pp. 2544–2550, 1963.

P. J. Werbos *et al.*, "Backpropagation through time: what it does and how to do it," *Proceedings of the IEEE*, vol. 78, no. 10, pp. 1550–1560, 1990.

D. E. Rumelhart and D. Zipser, "Feature discovery by competitive learning," *Cognitive science*, vol. 9, no. 1, pp. 75–112, 1985.

D. E. Rumelhart, G. E. Hinton, and R. J. Williams, "Neurocomputing: Foundations of research," ch. Learning Representations by Back-propagating Errors, pp. 696–699, Cambridge, MA, USA: MIT Press, 1988.

S. Haykin, *Neural Networks and Learning Machines*. No. v. 10 in Neural networks and learning machines, Prentice Hall, 2009.

R. H. Hahnloser, R. Sarpeshkar, M. A. Mahowald, R. J. Douglas, and H. S. Seung, “Digital selection and analogue amplification coexist in a cortex-inspired silicon circuit,” *Nature*, vol. 405, no. 6789, p. 947, 2000.

X. Glorot, A. Bordes, and Y. Bengio, “Deep sparse rectifier neural networks,” in *Proceedings of the fourteenth international conference on artificial intelligence and statistics*, pp. 315–323, 2011.

G. M. Shepherd, “The dendritic spine: a multifunctional integrative unit,” *Journal of neurophysiology*, vol. 75, no. 6, pp. 2197–2210, 1996.

F. Crick, “The recent excitement about neural networks.,” *Nature*, vol. 337, no. 6203, pp. 129–132, 1989.

D. G. Stork, “Is backpropagation biologically plausible,” in *International Joint Conference on Neural Networks*, vol. 2, pp. 241–246, IEEE Washington, DC, 1989.