Introduction to Machine Learning

Page Ranking and the Web

Andres Mendez-Vazquez

August 21, 2020

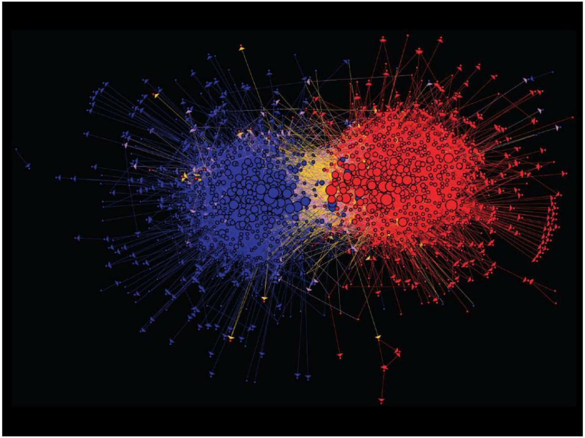
Outline

Graph Data: Social Networks

Facebook social graph

4-degrees of separation [Backstrom-Boldi-Rosa-Ugander-Vigna, 2011].

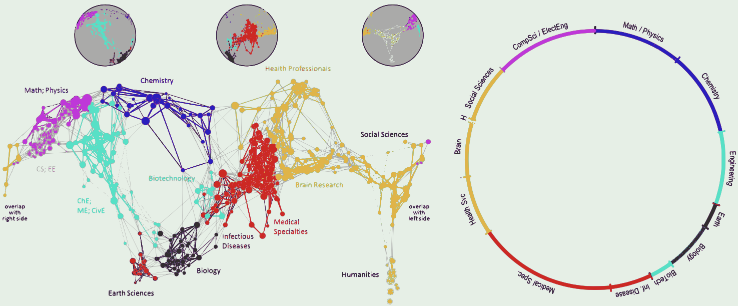
Graph Data: Media Networks



Connections between political blogs

Polarization of the network [Adamic-Glance, 2005].

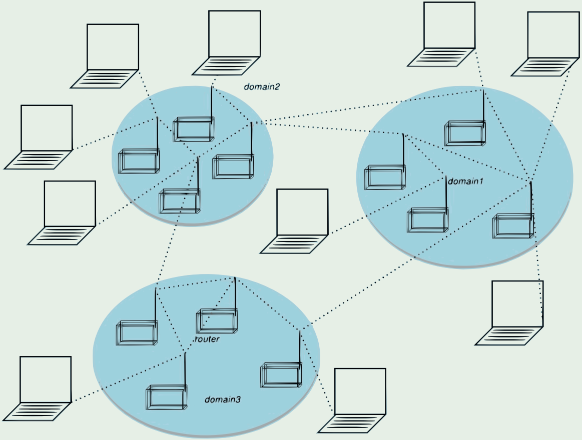
Graph Data: Information Nets



Citation networks and Maps of science

[Börner et al., 2012]

Graph Data: Communication Nets



Internet

Web as Graph

Web as a directed graph

- Nodes: Web-pages
- Edges: Hyperlinks

We can have the following Pages

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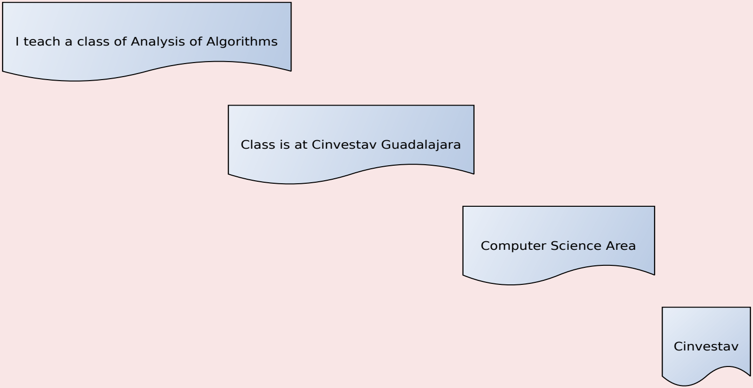
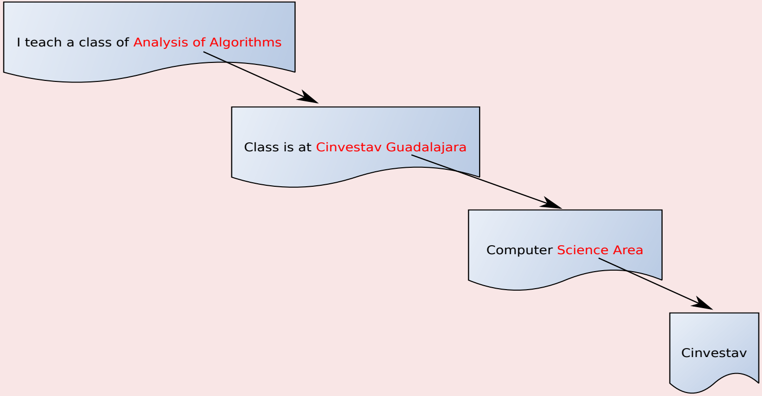
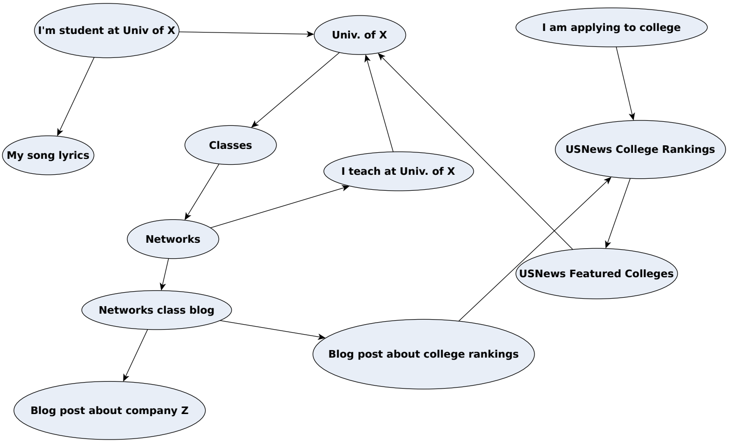


Web as Graph

Now Add the Edges



Another Example, A Semantic Web



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Broad Question

- How to organize the Web?

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First try

- Human curated Web directories
  - Yahoo, DMOZ, LookSmart

- Information Retrieval investigates: Find relevant docs in a small and trusted set
  - Newspaper articles, Patents, etc.
- But: the Web is huge, full of non-trustable documents, random things, web spam, etc.

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- Trick: Pages that actually know about newspapers might all be pointing to many newspapers

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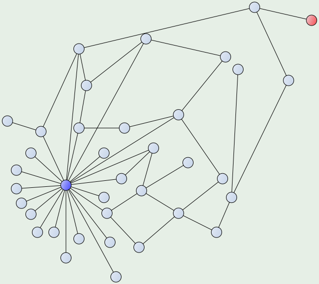
Ranking Nodes on the Graph

All web pages are not equally "important" : www.joe-schmoe.com vs. www.stanford.edu



Web-graph node connectivity

- There is large diversity in the web-graph node connectivity. Let's rank the pages by the link structure!

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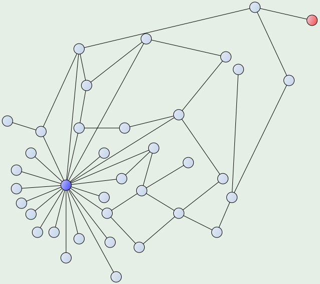

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We will cover the following Link Analysis approaches for computing importance of nodes in a graph

- Page Rank
- Hubs and Authorities (HITS)
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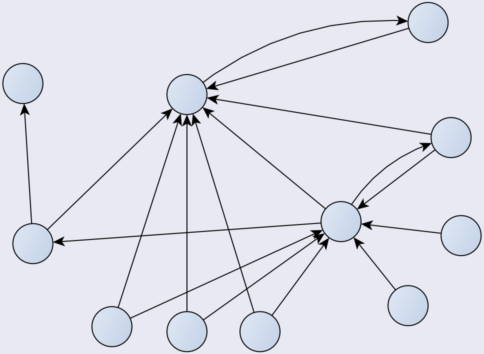

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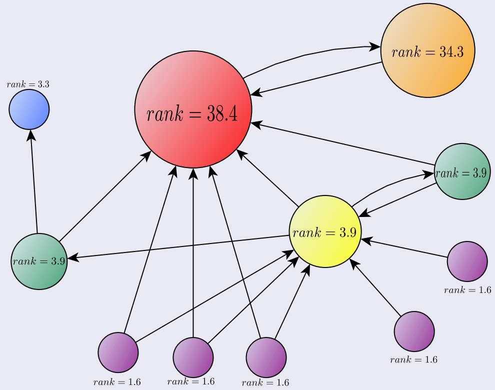

Simple Recursive Formulation

Link's vote

Each link's vote is proportional to the importance of its source page.

Out-links

If page j with importance r_j has n out-links, each link gets $\frac{r_j}{n}$ votes.

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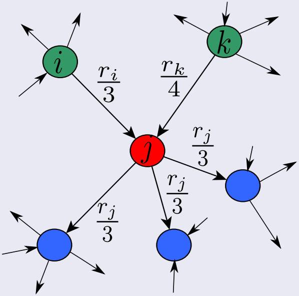

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Page Rank: The "Flow" Model

Voting

A "vote" from an important page is worth more

Importance

A page is important if it is pointed to by other important pages

Define a "rank" r_j for page j

$$r_j = \sum_{i \to j} \frac{r_i}{d_i}$$

d_i... out-degree of node i

Flow Equations

$$r_y = \frac{r_y}{2} + \frac{r_a}{2}$$

$$r_a = \frac{r_y}{2} + r_m$$

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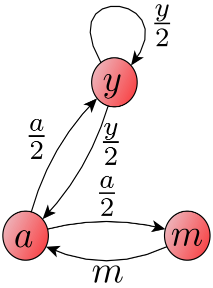
The Web in 1839



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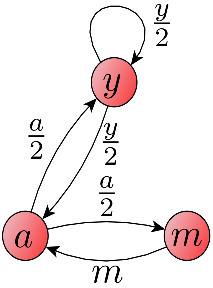

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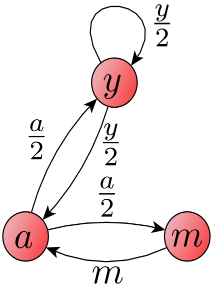

Solving the Flow Equations

Something Notable

- Flow equations: 3 equations, 3 unknowns, no constants:
  - No unique solution.
  - All solutions equivalent modulo the scale factor.
- Additional constraint forces uniqueness:
  - $r_y + r_a + r_m = 1$
    - Solution: $r_y = \frac{2}{5}$, $r_a = \frac{2}{5}$, $r_m = \frac{1}{5}$

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Stochastic adjacency matrix M

- Let page i has d_i out-links
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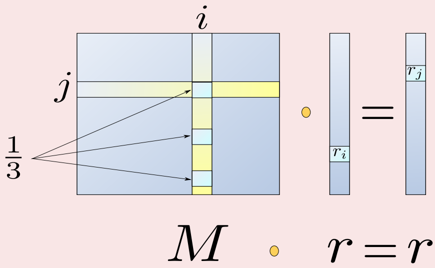

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Stochastic Matrices

Markov process

A stochastic process is called a Markov process when it has the Markov property:

$$P\left(X_{t_n}|X_{t_{n-1}}=x_{n-1},...X_{t_1}=x_1\right)=P\left(X_{t_n}|X_{t_{n-1}}=x_{n-1}\right) \quad (2)$$

Simply

The future path of a Markov process, given its current state and the past history before, depends only on the current state (not on how this state has been reached).

Thus (Quite an Oversimplification!!!)

A Markov process is characterized by the (one-step) transition probabilities:

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The future path of a Markov process, given its current state and the past history before, depends only on the current state (not on how this state has been reached).

Thus (Quite an Oversimplification!!!)

A Markov process is characterized by the (one-step) transition probabilities:

$$p_{i,j}=P\left(X_{t+1}=i|X_t=j\right) \quad (3)$$

Markov Chains

Definition (Oversimplified)

A Markov chain is a process X_t indexed by integers $t = 0, 1, \ldots$ such that the states X_t describe the chain at time t.

From here

The probability of a path $i_0, i_1, \ldots, i_n$ is

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P(X_0 = i_0)\, p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} \quad (4)$$

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The transition probabilities can be arranged as transition probability matrix $\boldsymbol{P} = (p_{i,j})$

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The column j contains the transition probabilities from state j to other states.

Since the system always goes to some state, the sum of the column probabilities is 1:

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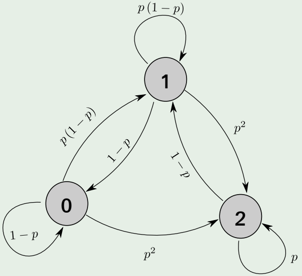
Stochastic Matrix

Definition

- A matrix with non-negative elements such that the sum of each column equals "ONE" is called a stochastic matrix.

Example

We have the following state machine



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$$P = \begin{pmatrix} 1-p & 1-p & 0 \\ p\left(1-p\right) & p\left(1-p\right) & 1-p \\ p^2 & p^2 & p \end{pmatrix} \quad (6)$$

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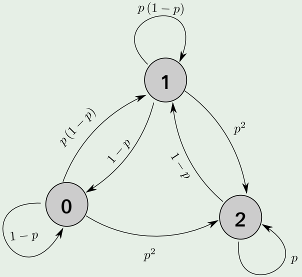

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To describe the transition in the Markov Chain

Let $\boldsymbol{p}_t \in \mathbb{R}^n$ is the distribution matrix of X_t at time t

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Here, we have Perron-Frobenius

Basic Definition

- A matrix is called positive if all its entries are positive.
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Let π a Perron-Frobenius right eigenvector of P with $\pi \geq 0$ and $1^T \pi = 1$.

Such that $P\pi = \pi$

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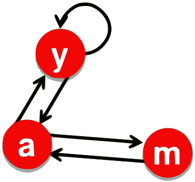

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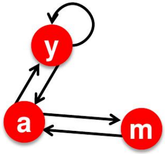

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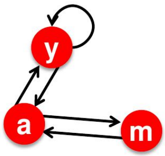

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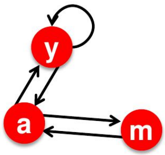

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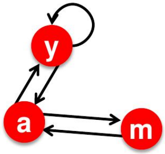

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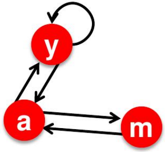

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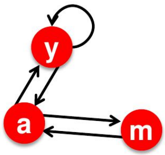

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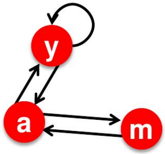

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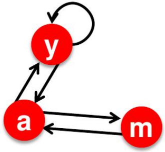

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Example

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Why Power Iteration works? (1)

Power iteration

- A method for finding dominant eigenvector (the vector corresponding to the largest eigenvalue)
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- Thus: $M^k r^{(0)} \approx C_1 \lambda_1^k x_1$
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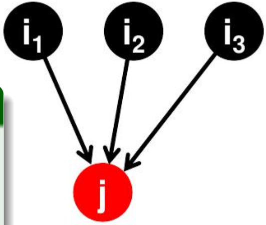

$$r_j = \sum_{i \to j} \frac{r_i}{d_{out}(i)}$$

Imagine a random web surfer

- At any time t, surfer is on some page i
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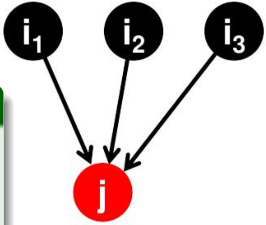

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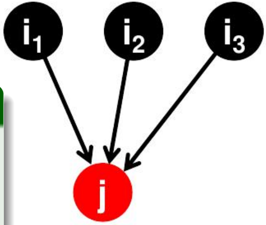

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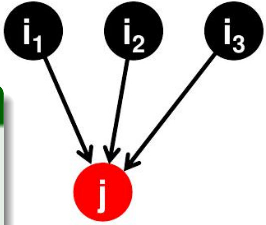

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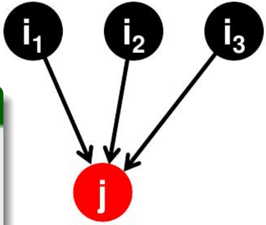

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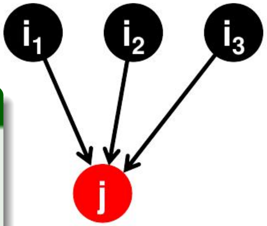

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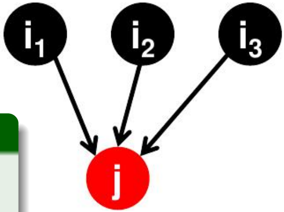
The Stationary Distribution



$$p(t+1) = M \cdot p(t)$$

Where is the surfer at time $t+1$?

- Follows a link uniformly at random $p(t+1) = M \cdot p(t)$.

Suppose

- Suppose the random walk reaches a state $p(t+1) = M \cdot p(t) = p(t)$ then $p(t)$ is stationary distribution of a random walk.

Our original rank vector

- Our original rank vector r satisfies $r = M \cdot r$
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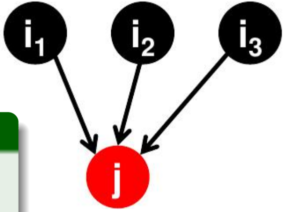

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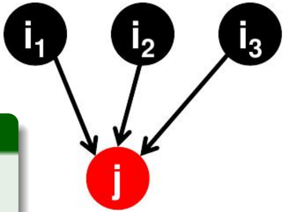

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$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{d_i}$$

Example

$$\begin{array}{c} r_a \\ r_b \end{array} = \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}$$

Iteration 0, 1, 2, ...

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Two problems:

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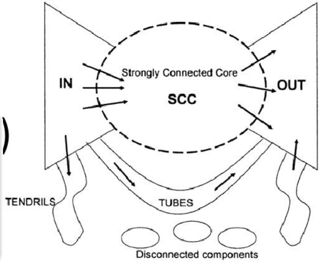
- Some pages are dead ends (have no out-links)
  - Such pages cause importance to "leak out"



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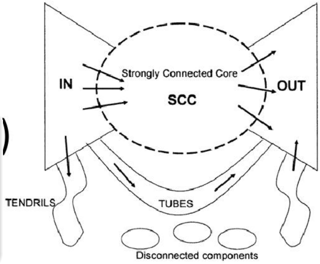

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- Spider traps (all out-links are within the group)
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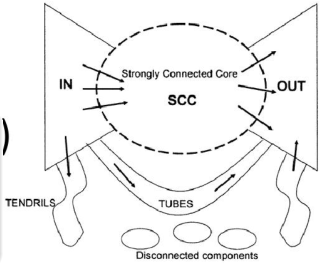

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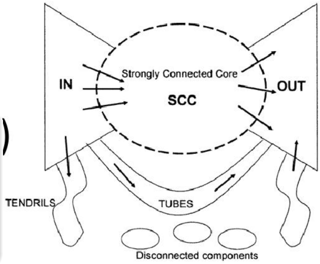

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Problems: Spider Traps

Power Iteration

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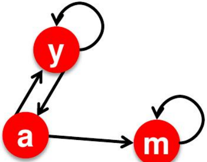
- $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - And iterate



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	1

$$r_y = r_y/2 + r_a/2$$
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Example

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{bmatrix} 1/3 & 2/6 & 3/12 & 5/24 & & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \cdots & 0 \\ 1/3 & 3/6 & 7/12 & 16/24 & & 1 \end{bmatrix}$$

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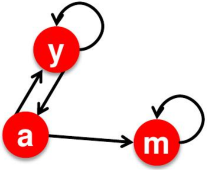

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The Google solution for spider traps

- At each time step, the random surfer has two options:
  - With prob. β, follow a link at random.
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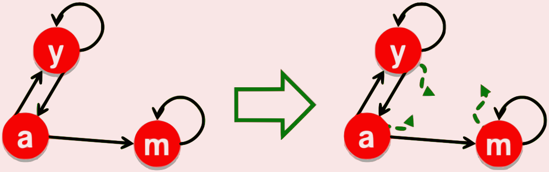
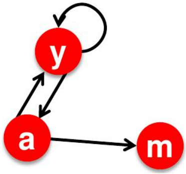
Surfer will teleport out of spider trap within a few time steps



Problem: Dead Ends

Power Iteration on the previous graph

- Set $r_j = 1$
- $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
  - And iterate



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y	½	½	0
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m	0	½	0

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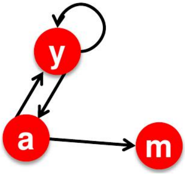

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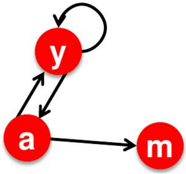

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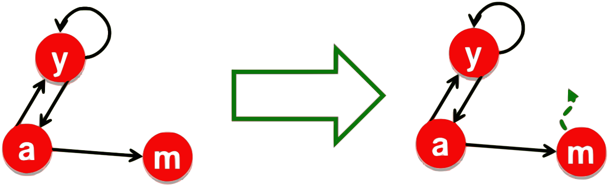
Solution: Always Teleport

Teleport

- Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly



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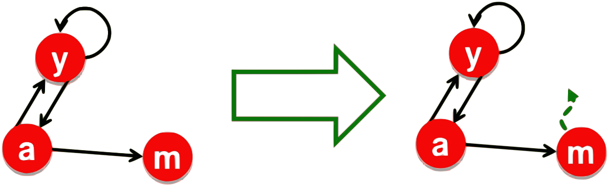

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Outline

Make M Stochastic

Stochastic:

- Every column sums to 1

$$r_y = r_y/2 + r_a/2 + r_m/3$$
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Make M Stochastic

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A possible solution

- Add green links

$$A = M + a\left(\frac{1}{n}e\right)^T$$

- $a_i = \begin{cases} 1 & \text{if node } i \text{ has out deg } 0 \\ 0 & \text{else} \end{cases}$
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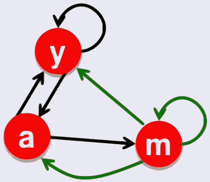

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Make M Aperiodic

Periodic

- A chain is periodic if there is $k > 1$ such that the interval between two visits to some state $\boldsymbol{s}$ is always a multiple of k.

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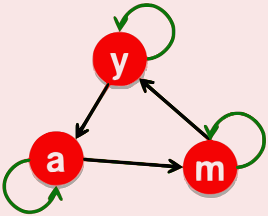

Make M Irreducible

Definition

- From any state, there is a non-zero probability of going from any one state to any another

A possible solution for a graph

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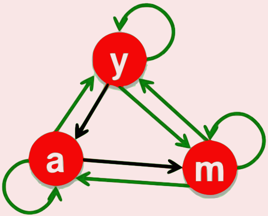

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Thus

Using the $S = M + a\left(\frac{1}{n}e^T\right)$ to handle nodes with out-degree 0

We can re-write the google matrix

$$A = \beta S + (1-\beta)\frac{1}{n}\mathbf{e}\cdot\mathbf{e^T} \tag{9}$$

Something Notable

The teleporting is random because the teleportation matrix $E = \frac{1}{n}\mathbf{e}\cdot\mathbf{e}^{\mathbf{T}}$ is uniform

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The surfer is equally likely, when teleporting, to jump to any page.

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There are several consequences of the primitivity adjustment

1. A is stochastic. It is the convex combination of the two stochastic matrices M and E.
2. A is irreducible. Every page is directly connected to every other page, so irreducibility is trivially enforced.
3. A is aperiodic. The self-loops ($A_{ii} > 0$ for all i) create aperiodicity.
4. A is primitive because $A^k > 0$ for some k. Implying that a a unique positive vector π exists, and the power method applied to A is guaranteed to converge to this vector.
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$$r^{(t+1)} = A \cdot r^{(t)}$$
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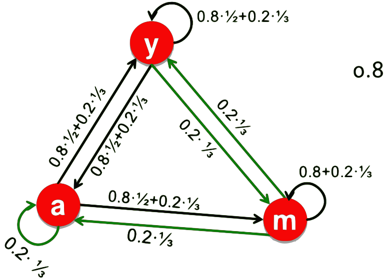
Example: Random Teleport ($\beta = 0.8$)



$$0.8 \underbrace{\begin{array}{|ccc|} \hline 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \\ \hline \end{array}}_{\textbf{M}} + 0.2 \underbrace{\begin{array}{|ccc|} \hline 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ \hline \end{array}}_{\textbf{1/n}\cdot\textbf{1}\cdot\textbf{1}^{\textbf{T}}}$$

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{array}{|ccc|} \hline 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \\ \hline \end{array}$$

A

$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{cccccc} 1/3 & 0.33 & 0.24 & 0.26 & & 7/33 \\ 1/3 & 0.20 & 0.20 & 0.18 & \ldots & 5/33 \\ 1/3 & 0.46 & 0.52 & 0.56 & & 21/33 \end{array}$$

Outline

Computing Page Rank

Key step is matrix-vector multiplication

$$r^{new} = A \cdot r^{old}$$

Easy

- Easy if we have enough main memory to hold A, r^{old}, r^{new}

However, if you have $N = 1$ billion pages

- We need 4 bytes for each entry (say)
- 2 billion entries for vectors, approx 8GB
- Matrix A has N^2 entries
  - 10^{18} is a large number!

$$\mathbf{A} = \beta \cdot \mathbf{M} + (1-\beta)\,[1/N]_{N\times N}$$

$$\boldsymbol{A} = 0.8 \begin{bmatrix} ½ & ½ & 0 \\ ½ & 0 & 0 \\ 0 & ½ & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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Suppose

- Suppose there are N pages.
- Consider page j, with d_j out-links.
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- We have $M_{ij} = 1/|d_j|$ when $j \to i$ and $M_{ij} = 0$ otherwise.

The random teleport is equivalent to

- Adding a teleport link from j to every other page and setting transition probability to $(1-\beta)/N$.
- Reducing the probability of following each out-link from $1/|d_j|$ to $\beta/|d_j|$.
- Equivalent: Tax each page a fraction $(1-\beta)$ of its score and redistribute evenly

Outline

Rearranging the Equation (1)

$A_{ij} = \beta M_{ij} + \frac{1-\beta}{N}$

$$\begin{aligned}
r &= A \cdot r \\
r_i &= \sum_{j=1}^{N} A_{ij} \cdot r_j \\
r_i &= \sum_{j=1}^{N} \left[\beta M_{ij} + \frac{1-\beta}{N} \right] \cdot r_j \\
&= \sum_{j=1}^{N} \beta M_{ij} \cdot r_j + \frac{1-\beta}{N} \sum_{j=1}^{N} r_j \\
&= \sum_{j=1}^{N} \beta M_{ij} \cdot r_j + \frac{1-\beta}{N} 1
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since $\sum_{j=1}^{N} r_j = 1$

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- 10 links per node, approx $10N$ entries

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Outline

Sparse Matrix Encoding

Encode sparse matrix using only non-zero entries

- Space proportional roughly to number of links
- Say $10N$, or $4 * 10 * 1$ billion $= 40$GB
- Still will not fit in memory, but will fit on disk

Source Node	Degree	Destination Node
0	3	1,5,6
1	4	17,64,113,117
2	2	12,23

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r^{new}: 0, 1, 2, 3, 4, 5, 6

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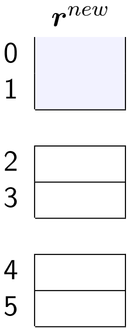
Block-based Update Algorithm



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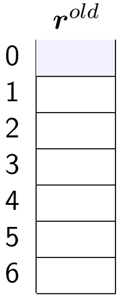

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Similar to nested-loop join in databases

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Block-Stripe Update Algorithm

$\mathbf{r}^{new}$

0
1

2
3

4
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2	2	1

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0
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3
4
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6

Block-Stripe Analysis

Break M into stripes

- Each stripe contains only destination nodes in the corresponding block of r^{new}

Some additional overhead per stripe

- But it is usually worth it

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