# Introduction to Machine Learning

## Convolutional Networks

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December 2, 2019

# Outline

# Outline



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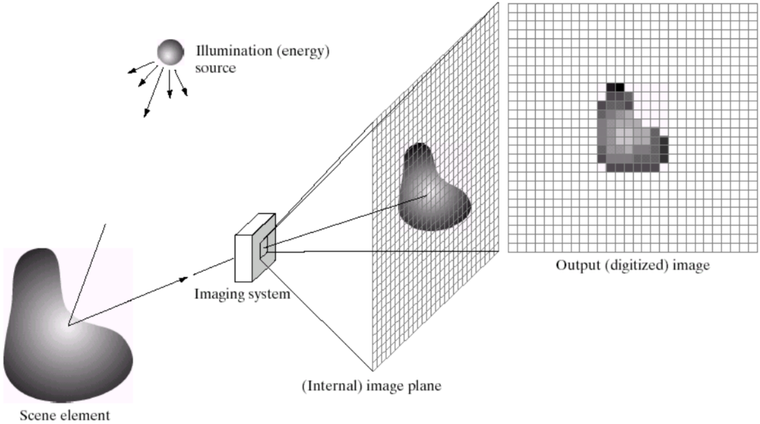
# Digital Images as pixels in a digitized matrix



# Further

**Pixel values typically represent**

- Gray levels, colours, heights, opacities etc

**Something Notable**

- Remember digitization implies that a digital image is an approximation of a real scene

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## Pixel values typically represent

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# Images

**Common image formats include**

- On sample/pixel per point (B&W or Grayscale)
- Three samples/pixel per point (Red, Green, and Blue)
- Four samples/pixel per point (Red, Green, Blue, and "Alpha")

# Therefore, we have the following process

## Low Level Process

| Input | Processes | Output |
|---|---|---|
| Image | Noise Removal | Improved Image |
| | Image Sharpening | |

# Example

**Edge Detection**

# Example

## Edge Detection

# Then

## Mid Level Process

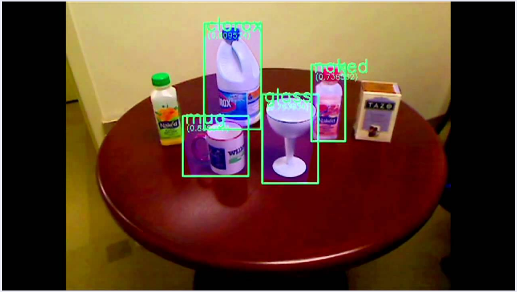
| Input | Processes | Output |
| --- | --- | --- |
| Image | Object Recognition | Attributes |
| | Segmentation | |

# Example

**Object Recognition**

# Example

## Object Recognition



# Therefore

### It would be nice to automatize all these processes

- We would solve a lot of headaches when setting up such process

### Why not to use the data sets

- By using a Neural Networks that replicates the process.

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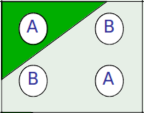
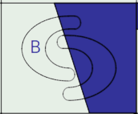
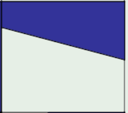
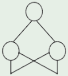
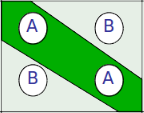
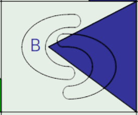
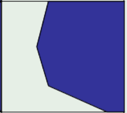
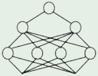
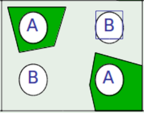
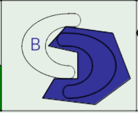
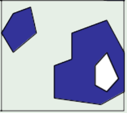
# Multilayer Neural Network Classification

## We have the following classification

| Structure | Types of Decision Regions | Exclusive-OR Problem | Classes with Meshed regions | Most General Region Shapes |
|---|---|---|---|---|
| Single-Layer  | Half Plane Bounded By Hyper plane |   |   |  |
| Two-Layer  | Convex Open Or Closed Regions |   |   |  |
| Three-Layer  | Arbitrary (Complexity Limited by No. of Nodes) |   |   |  |

# Outline



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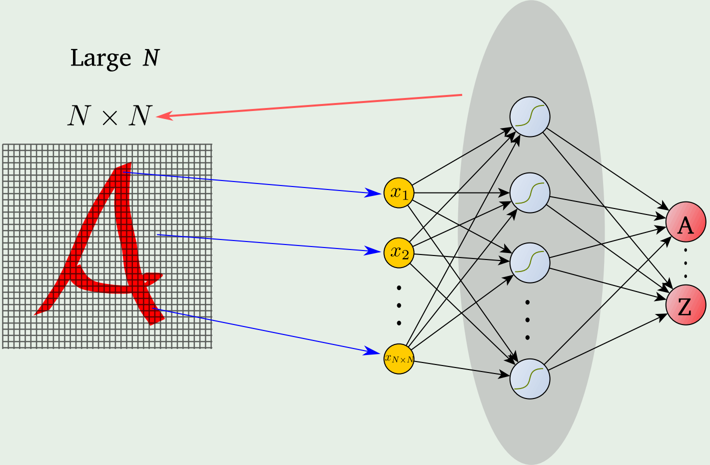
# Drawbacks of previous neural networks

## The number of trainable parameters becomes extremely large



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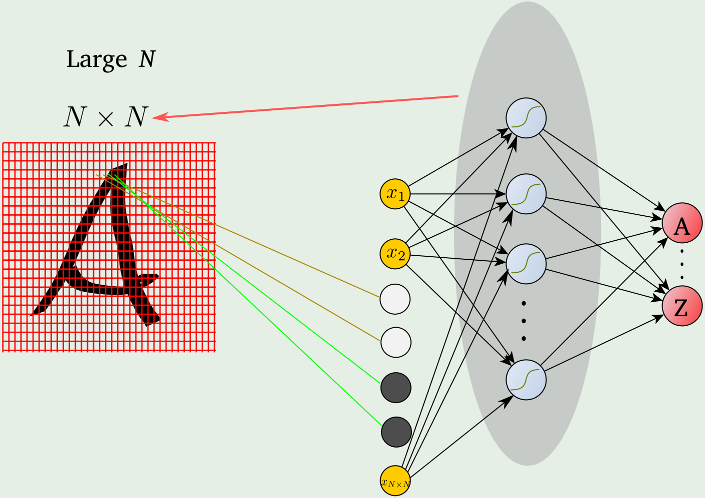
**In addition, little or no invariance to shifting, scaling, and other forms of distortion**



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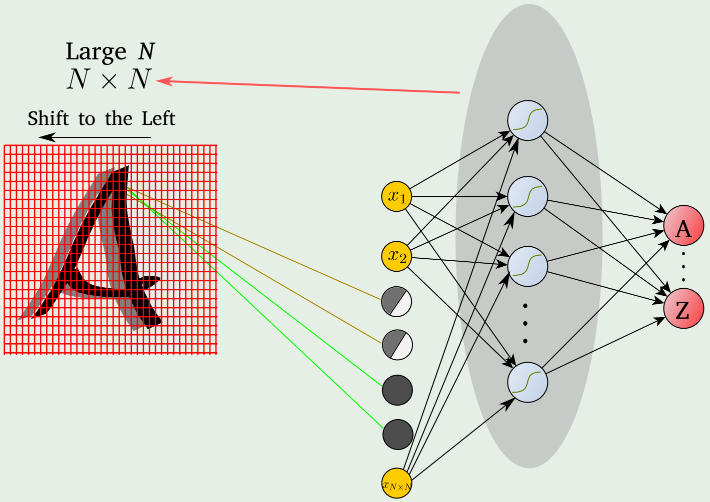
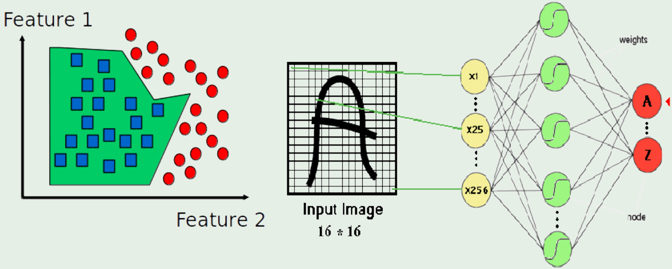


# Drawbacks of previous neural networks

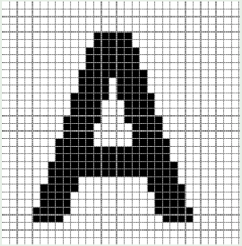
## The topology of the input data is completely ignored



# For Example

## We have

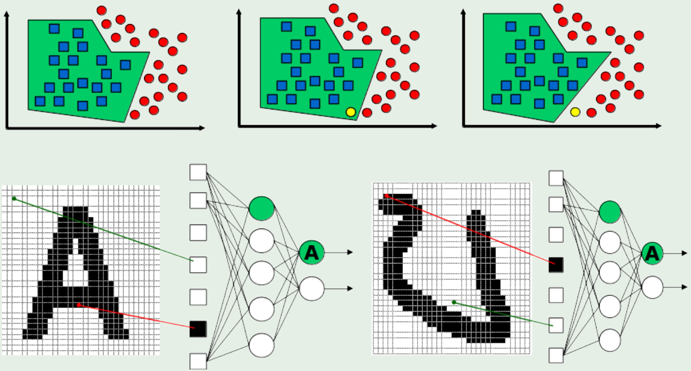
- Black and white patterns: $2^{32\times 32}=2^{1024}$
- Gray scale patterns: $256^{32\times 32}=256^{1024}$



**32 * 32 input image**

# For Example

## If we have an element that the network has never seen



# Possible Solution

**We can minimize this drawbacks by getting**

Fully connected network of sufficient size can produce outputs that are invariant with respect to such variations.

**Problem!!!**

- Training time
- Network size
- Free parameters

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D. Hubel and T. Wiesel (1959, 1962, Nobel Prize 1981)

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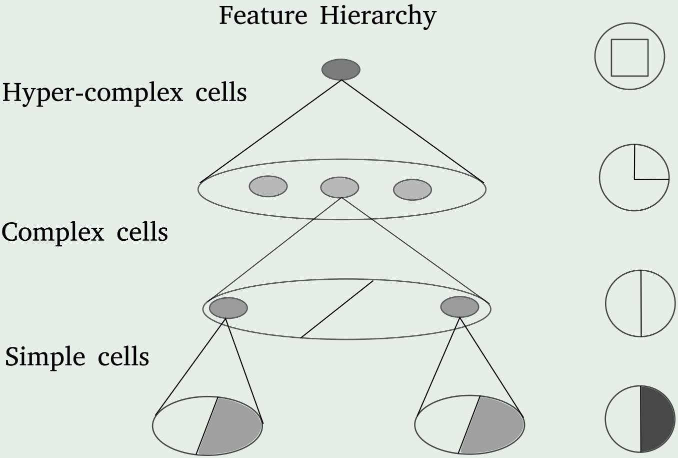
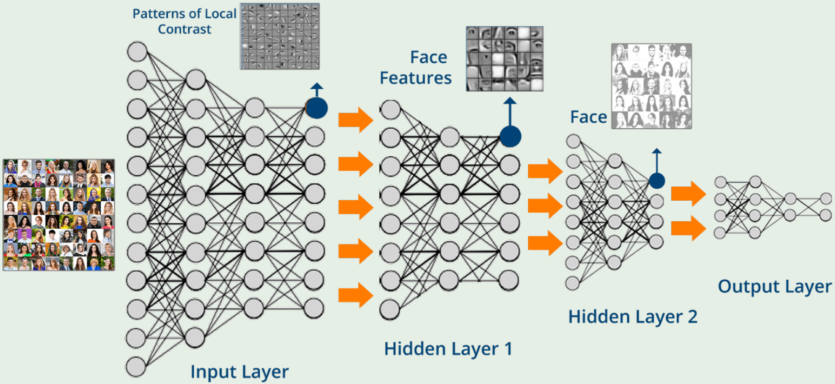


# History

## Convolutional Neural Networks (CNN) were invented by

In 1989, Yann LeCun and Yoshua Bengio introduced the concept of Convolutional Neural networks.



# About CNN's

### Something Notable

CNN's Were neurobiologically motivated by the findings of locally sensitive and orientation-selective nerve cells in the visual cortex.

### In addition

They designed a network structure that implicitly extracts relevant features.

### Properties

Convolutional Neural Networks are a special kind of multilayer neural networks.

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## In addition

- CNN is a feed-forward network that can extract topological properties from an image.
- Like almost every other neural networks they are trained with a version of the back-propagation algorithm.
- Convolutional Neural Networks are designed to recognize visual patterns directly from pixel images with minimal preprocessing.
- They can recognize patterns with extreme variability.

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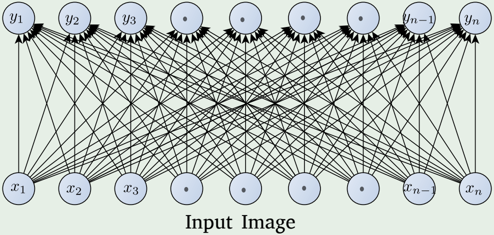
# Local Connectivity

## We have the following idea

Instead of using a full connectivity...




## We would have something like this

$$y_i = f\left(\sum_{i=1}^{n} w_i x_i\right) \quad (1)$$

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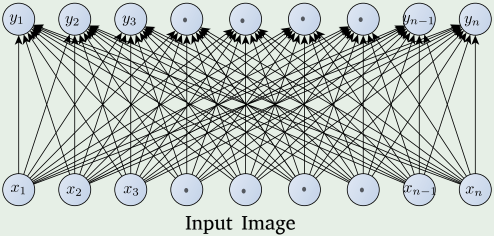


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# Local Connectivity

**We decide only to connect the neurons in a local way**

- Each hidden unit is connected only to a subregion (patch) of the input image.
- It is connected to all channels:
  - 1 if gray scale
  - 3 in the RGB case

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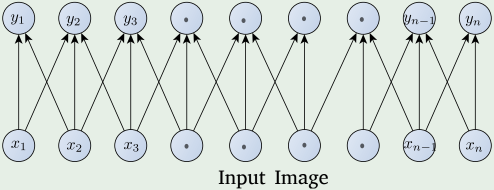


## Then, our formula changes

$$y_i = f\left(\sum_{i \in L_p} w_i x_i\right) \quad (2)$$

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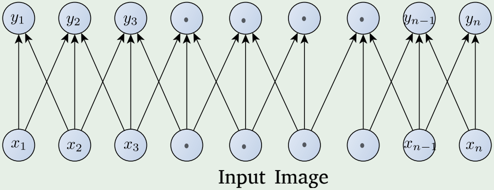


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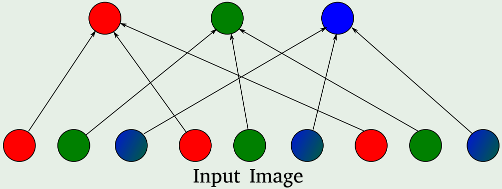
## In the case of the 3 channels

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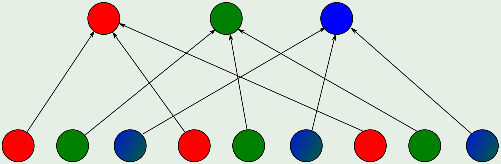


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# Solving the following problems...

**First**

Fully connected hidden layer would have an unmanageable number of parameters

**Second**

Computing the linear activation of the hidden units would have been quite expensive

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# How this looks in the image...

## We have

Receptive Field

# Outline



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# Parameter Sharing

**Second Idea**

Share matrix of parameters across certain units.

**These units are organized into**

- The same feature "map"
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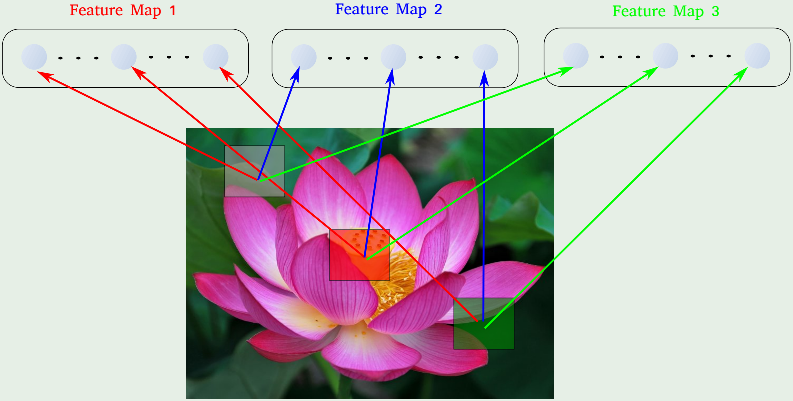

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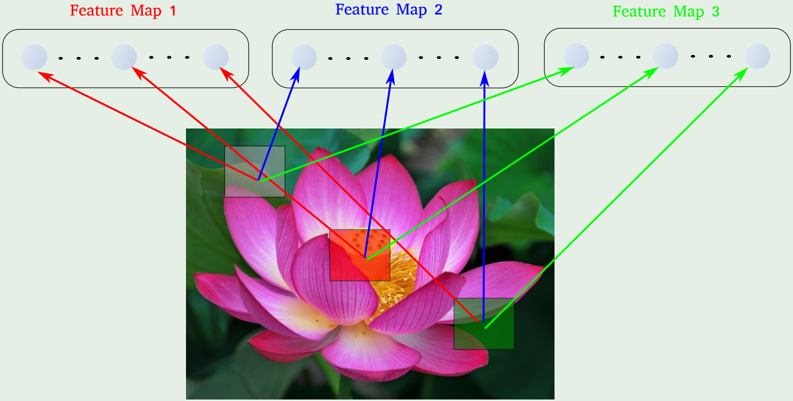

# Now, in our notation

## We have a collection of matrices representing this connectivity

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**An now why the name of convolution**

Yes!!! The definition is coming now.

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The image can now be represented as a matrix of integer values, $f : [a, b] \times [c, d] \rightarrow I$

$$
\begin{array}{c c}
 & j \longrightarrow \\
i \downarrow & \begin{bmatrix}
79 & 5 & 6 & 90 & 12 & 34 & 2 & 1 \\
8 & 90 & 12 & 34 & 26 & 78 & 34 & 5 \\
8 & 1 & 3 & 90 & 12 & 34 & 11 & 61 \\
77 & 90 & 12 & 34 & 200 & 2 & 9 & 45 \\
1 & 3 & 90 & 12 & 20 & 1 & 6 & 23
\end{bmatrix}
\end{array}
$$

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$$

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# We can see the coordinate of $f$ as follows

## We have the following

$$f = \begin{pmatrix} f_{-n,-n} & f_{-n,-n+1} & \cdots & f_{-n,(n-1)} & f_{-n,n} \\ \vdots & \ddots & \vdots & \iddots & \vdots \\ \vdots & \cdots & f_{0,0} & \cdots & \vdots \\ \vdots & \iddots & \vdots & \ddots & \vdots \\ f_{n\times -n} & f_{n\times -n+1} & \cdots & f_{n\times(n-1)} & f_{n,n} \end{pmatrix} \quad (4)$$

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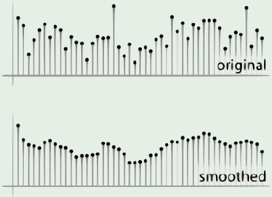
# Many times we want to eliminate noise in a image

## By using for example a moving average




## This last moving average can be seen as

$$(f * g)(i) = \sum_{j=-n}^{n} f(j)\, g(i-j) = \frac{1}{N} \sum_{j=m}^{-m} f(j) \quad (5)$$

With $f(j)$ representing the value of the pixel at position $i$,

$$g(h) = \begin{cases} \frac{1}{N} & \text{if } h \in \{-m, -m+1, ..., 1, 0, 1, ..., m-1, m\} \\ 0 & \text{else} \end{cases}$$

with $0 < m < n$.

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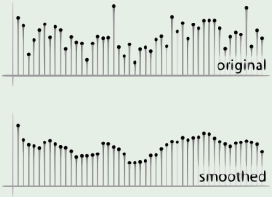


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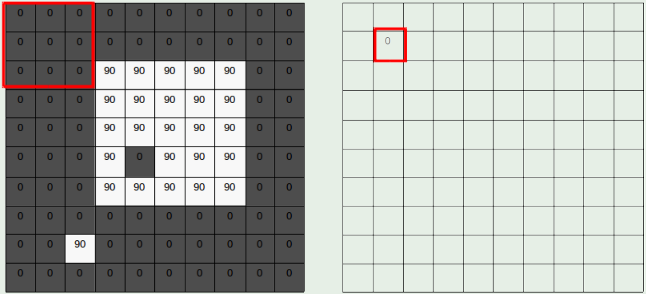
# This can be generalized into the 2D images

## Left $f$ and Right $f * g$



# This can be generalized into the 2D images

## Left $f$ and Right $f * g$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 0 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 0 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
| | 0 | 10 | 20 | | | | | | |
| | | | | | | | | | |
| | | | | | | | | | |
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| | | | | | | | | | |
| | | | | | | | | | |
| | | | | | | | | | |
| | | | | | | | | | |

# This can be generalized into the 2D images

## Left $f$ and Right $f * g$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 0 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 90 | 90 | 90 | 90 | 90 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 90 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |
| | 0 | 10 | 20 | 30 | 30 | 30 | 20 | 10 | |
| | 0 | 20 | 40 | 60 | 60 | 60 | 40 | 20 | |
| | 0 | 30 | 60 | 90 | 90 | 90 | 60 | 30 | |
| | 0 | 30 | 50 | 80 | 80 | 90 | 60 | 30 | |
| | 0 | 30 | 50 | 80 | 80 | 90 | 60 | 30 | |
| | 0 | 20 | 30 | 50 | 50 | 60 | 40 | 20 | |
| | 10 | 20 | 30 | 30 | 30 | 30 | 20 | 10 | |
| | 10 | 10 | 10 | 0 | 0 | 0 | 0 | 0 | |
| | | | | | | | | | |

# Moving average in 2D

## Basically in 2D

We have that we can define different types of filter using the idea of weighted average

$$(f * g)(i,j) = \sum_{k=n}^{-n} \sum_{l=-n}^{n} f(k,l) \times g(i-k, j-l) \tag{6}$$

## What is this weight matrix also called a kernel of $3 \times 3$ moving average

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{"The Box Filter"} \tag{7}$$

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# Outline



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# Convolution

## Definition

Let $f : [a, b] \times [c, d] \rightarrow I$ be the image and $g : [e, f] \times [h, i] \rightarrow V$ be the kernel. The output of convolving $f$ with $g$, denoted $f * g$ is

$$(f * g)[x, y] = \sum_{k=-n}^{n} \sum_{l=-n}^{n} f(k, l) g(x - k, y - l) \tag{8}$$

- The Flipped Kernel

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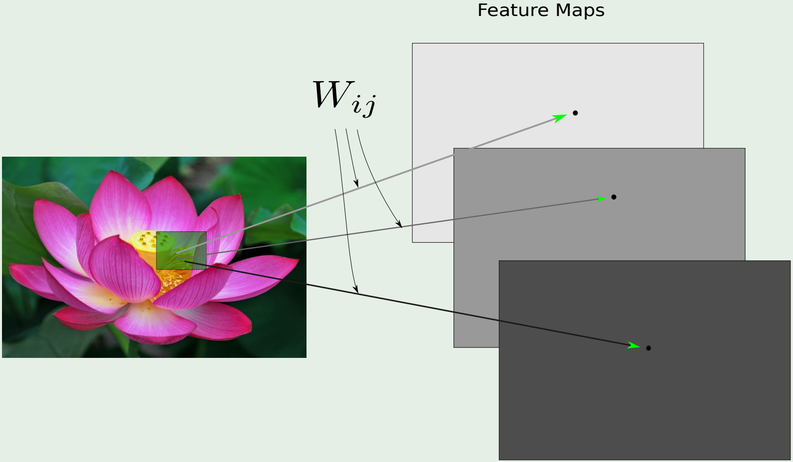
# Back on the Convolutional Architecture

## We have then something like this



# Thus

## Each Feature Map forms a 2D grid of features

That can be computed with a discrete convolution (*) of a kernel matrix $k_{ij}$ which is the hidden weights matrix $W_{ij}$ with rows and columns with its rows and columns flipped.

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- $x_i$ is the $i$th channel of input.
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**Let layer $l$ be a Convolutional Layer**

Then, the input of layer $l$ comprises $m_1^{(l-1)}$ feature maps from the previous layer.

**Each input layer has a size of $m_2^{(l-1)} \times m_3^{(l-1)}$**

In the case where $l = 1$, the input is a single image $I$ consisting of one or more channels.

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**We have that**

- A Convolutional Neural Network (CNN) directly accepts raw images as input.

**Thus, their importance when training discrete filters**

- Instead of assuming a certain comprehension of Computer Vision, one could think this is as a Silver Bullet.

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## We have the following

- $Y_j^{(l)}$ is a matrix representing the $l$ layer and $j^{th}$ feature map.

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**Given a specific layer $l$, we have that $i^{th}$ feature map in such layer equal to**

$$Y_i^{(l)} = B_i^{(l)} + \sum_{j=1}^{m_1^{(l-1)}} K_{ij}^{(l)} * Y_j^{(l-1)} \tag{10}$$

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## Something Notable

- $m_2^{(l)}$ and $m_3^{(l)}$ are influenced by border effects.
- Therefore, the output feature maps when the convolutional sum is defined properly have size

$$m_2^{(l)} = m_2^{(l-1)} - 2h_1^{(l)}$$
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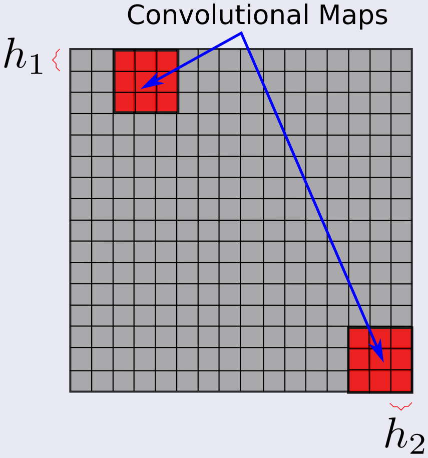
# Why?

## Example



# Special Case

**When $l = 1$**

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# Thus

## We have

Each feature map $Y_i^{(l)}$ in layer $l$ consists of $m_1^{(l)} \cdot m_2^{(l)}$ units arranged in a two dimensional array.

## Thus, the unit at position $(r, s)$ computes

$$\left(Y_i^{(l)}\right)_{r,s} = \left(B_i^{(l)}\right)_{r,s} + \sum_{j=1}^{m_1^{(l-1)}} \left(K_{ij}^{(l)} * Y_j^{(l-1)}\right)_{r,s}$$

$$= \left(B_i^{(l)}\right)_{r,s} + \sum_{j=1}^{m_1^{(l-1)}} \sum_{k=-h_1^{(l)}}^{h_1^{(l)}} \sum_{t=-h_2^{(l)}}^{h_2^{(l)}} \left(K_{ij}^{(l)}\right)_{k,t} \left(Y_j^{(l-1)}\right)_{r+k,s+t}$$

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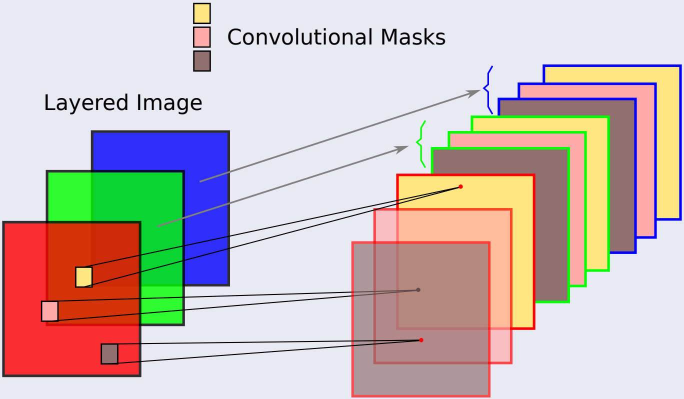
# Example

## A Convolutional Layer against a RGB Image using three masks/filters



# Outline

# As in Multilayer Perceptron

## We use a non-linearity

- However, there is a drawback when using Back-Propagation under a sigmoid function

$$s(x) = \frac{1}{1+e^{-x}}$$

## Because if we imagine a Convolutional Network as a series of layer functions $f_i$

$$y(A) = f_t \circ f_{t-1} \circ \cdots \circ f_2 \circ f_1(A)$$

With $f_t$ is the last layer.

## Therefore, we finish with a sequence of derivatives

$$\frac{\partial y(A)}{\partial w_{1i}} = \frac{\partial f_t(f_{t-1})}{\partial f_{t-1}} \cdot \frac{\partial f_{t-1}(f_{t-2})}{\partial f_{t-2}} \cdot \cdots \cdot \frac{\partial f_2(f_1)}{\partial f_2} \cdot \frac{\partial f_1(A)}{\partial w_{1i}}$$

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## Given the commutativity of the product

- You could put together the derivative of the sigmoid's

$$f(x) = \frac{ds(x)}{dx} = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}$$

## Therefore, deriving again

$$\frac{df(x)}{dx} = -\frac{e^{-x}}{\left(1+e^{-x}\right)^2} + \frac{2\left(e^{-x}\right)^2}{\left(1+e^{-x}\right)^3}$$

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## Therefore, Given a **Deep** Convolutional Network

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$$\lim_{k\to\infty}\left(\frac{ds\left(x\right)}{dx}\right)^{k} = \lim_{k\to\infty}\left(0.25\right)^{k} \to 0$$

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$$f(x) = x^{+} = \max(0, x)$$

## It is called ReLu or Rectifier

With a smooth approximation (Softplus function)

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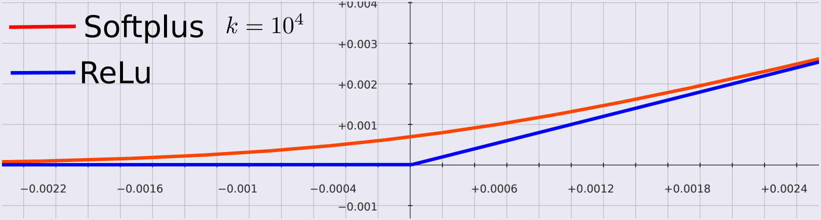

# Non-Linearity Layer

**If layer l is a non-linearity layer**

Its input is given by $m_1^{(l)}$ feature maps.

**What about the output**

Its output comprises again $m_1^{(l)} = m_1^{(l-1)}$ feature maps

**Each of them of size**

$$m_2^{(l-1)} \times m_3^{(l-1)} \quad (11)$$

With $m_2^{(l)} = m_2^{(l-1)}$ and $m_3^{(l)} = m_3^{(l-1)}$.

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# Thus

**With the final output**

$$Y_i^{(l)} = f\left(Y_i^{(l-1)}\right) \tag{12}$$

**Where**

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# Outline



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# Rectification Layer, $R_{abs}$

## Now a rectification layer

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$$f(x) = \frac{\ln\left(1 + e^{kx}\right)}{k}$$

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**Remark**

- Rectification could be included in the non-linearity layer.
- But also it can be seen as an independent layer.

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Experiments show that rectification plays a central role in achieving good performance.

**You can find this in**

K. Jarrett, K. Kavukcuogl, M. Ranzato, and Y. LeCun. What is the best multi-stage architecture for object recognition? In Computer Vision, International Conference on, pages 2146–2153, 2009.

**Remark**

- Rectification could be included in the non-linearity layer.
- But also it can be seen as an independent layer.

# Given that we are using Backpropagation

**We need a soft approximation to $f(x) = |x|$**

For this, we have

$$\frac{\partial f}{\partial x} = \text{sgn}(x)$$

- When $x \neq 0$. Why?

**We can use the following approximation**

$$\text{sgn}(x) = 2\left(\frac{\exp\{kx\}}{1+\exp\{kx\}}\right) - 1$$

**Therefore, we have by integration and working the $C$**

$$f(x) = \frac{2}{k}\ln\left(1+\exp\{kx\}\right) - x - \frac{2}{k}\ln(2)$$

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# We get the following situation

## Something Notable

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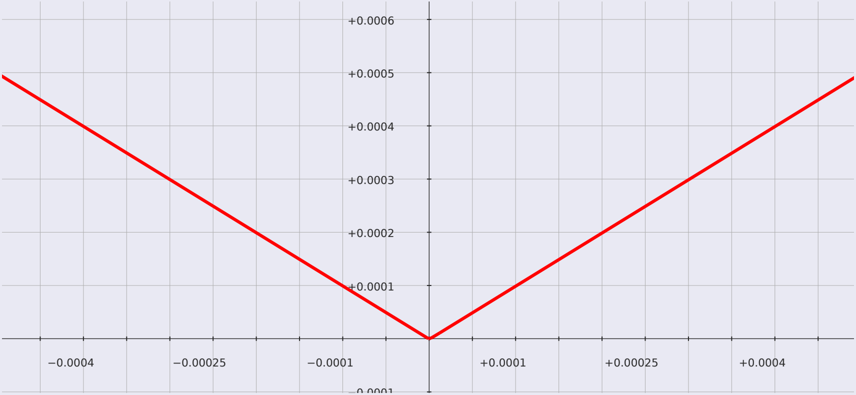

# Outline



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# Normalizing

## Contrast normalization layer

The task of a local contrast normalization layer:

- To enforce local competitiveness between adjacent units within a feature map.
- To enforce competitiveness units at the same spatial location.

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# Subtractive Normalization

**Given $m_1^{(l-1)}$ feature maps of size $m_2^{(l-1)} \times m_3^{(l-1)}$**

The output of layer $l$ comprises $m_1^{(l)} = m_1^{(l-1)}$ feature maps unchanged in size.

**With the operation**

$$Y_i^{(l)} = Y_i^{(l-1)} - \sum_{j=1}^{m_1^{(l-1)}} K_{G(\sigma)} * Y_j^{(l-1)} \quad (15)$$

**With**

$$\left(K_{G(\sigma)}\right)_{r,s} = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{r^2+s^2}{2\sigma^2}\right\} \quad (16)$$

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# Brightness Normalization

**An alternative is to normalize the brightness in combination with the rectified linear units**

$$\left(Y_i^{(l)}\right)_{r,s} = \frac{\left(Y_i^{(l-1)}\right)_{r,s}}{\left(\kappa + \lambda \sum_{j=1}^{m_1^{(l-1)}} \left(Y_j^{(l-1)}\right)_{r,s}^2\right)^{\mu}} \tag{17}$$

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# Outline



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# Subsampling Layer

## Motivation

The motivation of subsampling the feature maps obtained by previous layers is robustness to noise and distortions.

## How?

- Normally, in traditional Convolutional Networks subsampling this is done by applying skipping factors!!!
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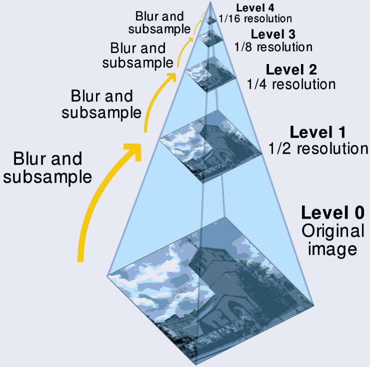
# Sub-sampling

## The subsampling layer

- It seems to be acting as the well know sub-sampling pyramid



# How is subsampling implemented?

## We know that Image Pyramids

They were designed to find:

1. filter-based representations to decompose images into information at multiple scales,
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# Projection Vectors

## Let $I \in \mathbb{R}^N$ an image

And a projection transformation such that

$$\boldsymbol{a} = PI$$

## Where

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_0 & \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{M-1} \end{bmatrix} \in \mathbb{R}^M$$

- The transformation coefficients...

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**When $M = N$**

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# Therefore

**We have that we can build a series of subsampled images**

$$\left\{ \begin{array}{cccc} I_0 & I_1 & \cdots & I_T \end{array} \right\}$$

**Usually constructed with a separable 1D kernel $h$**

$$I_{k+1} = PI_k = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\text{down-sampling}} \underbrace{\begin{pmatrix} \ddots & & & & & \\ - & h & - & & & \\ & - & h & - & & \\ & & - & h & - & \\ & & & & & \ddots \end{pmatrix}}_{\text{conv toplitz matrix}} I_k$$

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# There are also other ways of doing this

**subsampling can be done using so called skipping factors**

$$s_1^{(l)} \text{ and } s_2^{(l)}$$

**The basic idea is to skip a fixed number of pixels**

Therefore the size of the output feature map is given by

$$m_2^{(l)} = \frac{m_2^{(l-1)} - 2h_1^{(l)}}{s_1^{(l)} + 1} \text{ and } m_3^{(l)} = \frac{m_3^{(l-1)} - 2h_2^{(l)}}{s_2^{(l)} + 1}$$

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# What is Pooling?

**Pooling**

**Spatial pooling is way to compute image representation based on encoded local features.**

# Pooling

**Let $l$ be a pooling layer**

Its output comprises $m_1^{(l)} = m_1^{(l-1)}$ feature maps of reduced size.

**Pooling Operation**

It operates by placing windows at non-overlapping positions in each feature map and keeping one value per window such that the feature maps are subsampled.

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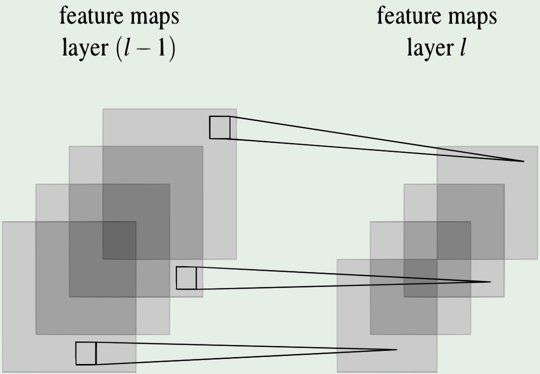
# Example

**If layer l is a pooling and subsampling layer and given $m_1^{(l-1)} = 4$ feature maps**



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**In the previous example**

All feature maps are pooled and subsampled individually.

**Each unit**

In one of the $m_1^{(l)} = 4$ output feature maps represents the average or the maximum within a fixed window of the corresponding feature map in layer $(l-1)$.

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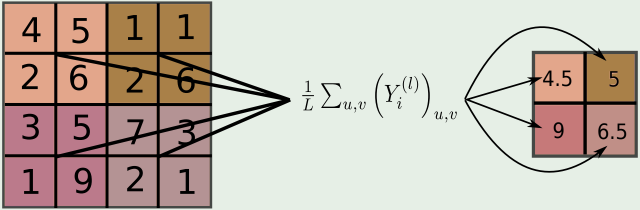
# We distinguish two types of pooling

## Average pooling

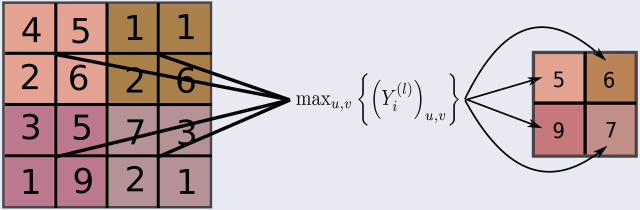
When using a boxcar filter, the operation is called average pooling and the layer denoted by $P_A$.



# We distinguish two types of pooling

## Max pooling

For max pooling, the maximum value of each window is taken. The layer is denoted by $P_M$.



# Outline



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# Fully Connected Layer

## If a layer $l$ is a fully connected layer

If layer $(l-1)$ is a fully connected layer, use the equation to compute the output of $i^{th}$ unit at layer $l$:

$$z_i^{(l)} = \sum_{k=0}^{m^{(l)}} w_{i,k}^{(l)} y_k^{(l)} \text{ thus } y_i^{(l)} = f\left(z_i^{(l)}\right)$$

## Otherwise

Layer $l$ expects $m_1^{(l-1)}$ feature maps of size $m_2^{(l-1)} \times m_3^{(l-1)}$ as input.

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# Then

**Thus, the $i^{\text{th}}$ unit in layer $l$ computes**

$$y_i^{(l)} = f\left(z_i^{(l)}\right)$$

$$z_i^{(l)} = \sum_{j=1}^{m_1^{(l-1)}} \sum_{r=1}^{m_2^{(l-1)}} \sum_{s=1}^{m_3^{(l-1)}} w_{i,j,r,s}^{(l)} \left(Y_j^{(l-1)}\right)_{r,s}$$

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# Here

## Where $w_{i,j,r,s}^{(l)}$

- It denotes the weight connecting the unit at position $(r, s)$ in the $j^{th}$ feature map of layer $(l-1)$ and the $i^{th}$ unit in layer $l$.

## Something Notable

- In practice, Convolutional Layers are used to learn a feature hierarchy and one or more fully connected layers are used for classification purposes based on the computed features.

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# Basically

## We can use a loss function at the output of such layer

$$L\left(\boldsymbol{W}\right)=\sum_{n=1}^{N}E_{n}\left(\boldsymbol{W}\right)=\sum_{n=1}^{N}\sum_{k=1}^{K}\left(y_{nk}^{(l)}-t_{nk}\right)^{2}\text{ (Sum of Squared Error)}$$

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## Assuming $W$ the tensor used to represent all the possible weights

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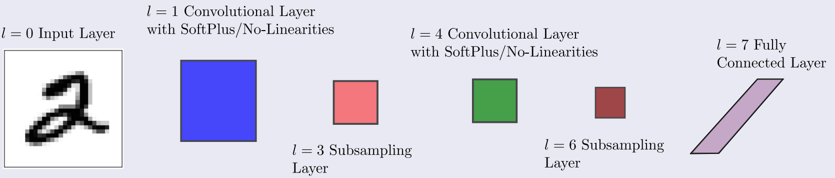
# Outline

# We have the following Architecture

## Simplified Architecture by Jean LeCun "Backpropagation applied to handwritten zip code recognition"



# Therefore, we have

## Layer $l = 1$

- This Layer is using a Softplus $f$ with 1 channels $j = 1$ Black and White

$$f\left[\left(Y_1^{(1)}\right)_{r,s}\right] = f\left[\left(B_1^{(l)}\right)_{r,s} + \sum_{k=-h_1^{(1)}}^{h_1^{(1)}} \sum_{t=-h_2^{(1)}}^{h_2^{(1)}} \left(K_{ij}^{(1)}\right)_{k,t} \left(Y_1^{(0)}\right)_{r+k,s+t}\right]$$

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# Now

**We have the $l=2$ subsampling for each coordinate**

$$Y_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} f\left[\left(Y_1^{(1)}\right)\right] \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^T$$

# Then, you repeat the previous

**Thus we obtain a reduced convoluted version $Y_1^{(6)}$ of the $Y_1^{(4)}$ convolution and subsampling**

- Thus, we use those as inputs for the fully connected layer of input.

**Now assuming a single $k=1$ neuron**

$$y_1^{(7)} = f\left(z_1^{(7)}\right)$$

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# We have

**That our final cost function is equal to**

$$L(\boldsymbol{t}) = \frac{1}{2}\left(y_1^{(7)} - t_1^{(7)}\right)^2$$

# Outline



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# After collecting all input/output

## Therefore

- We have using sum of squared errors (loss function):

$$\min_{\boldsymbol{W}} H(\boldsymbol{W}) = \frac{1}{2}\left(y_1^{(7)} - t_1^{(7)}\right)^2$$

## Therefore, we can obtain

$$\frac{\partial H(\boldsymbol{W})}{\partial w_{1,r,s}^{(7)}} = \frac{1}{2} \times \frac{\partial \left(y_1^{(7)} - t_1^{(7)}\right)^2}{\partial w_{1,r,s}^{(7)}}$$

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$$\frac{\partial f\left(z_1^{(7)}\right)}{\partial z_1^{(7)}} = \frac{e^{kz_1^{(7)}}}{\left(1 + e^{kz_1^{(7)}}\right)}$$

## Finally

$$\frac{\partial z_1^{(7)}}{\partial w_{1,r,s}^{(7)}} = \left(Y_1^{(6)}\right)_{r,s}$$

# Now

**Given the pooling**

$$Y_1^{(6)} = Sf\left[\left(Y_1^{(4)}\right)\right]S^T$$

**We have that**

$$\left(Y_1^{(4)}\right)_{r,s} = \left(B_1^{(4)}\right)_{r,s} + \sum_{k=-h_1^{(l)}}^{h_1^{(l)}} \sum_{l=-h_2^{(l)}}^{h_2^{(l)}} \left(K_{11}^{(4)}\right)_{k,l} \left(Y^{(3)}\right)_{r+k,s+l}$$

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# Therefore

**We have then**

$$\frac{\partial H\left(\boldsymbol{W}\right)}{\partial\left(K_{11}^{(4)}\right)_{k,t}}=\frac{1}{2}\times\frac{\partial\left(y_1^{(7)}-t_1\right)^2}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

**We have the following chain of derivations**

$$\frac{\partial H\left(\boldsymbol{W}\right)}{\partial\left(K_{11}^{(4)}\right)_{k,t}}=\left(y_i^{(l)}-t_i\right)\frac{\partial f\left(z_i^{(7)}\right)}{\partial z_i^{(7)}}\times\frac{\partial z_i^{(7)}}{\partial\left(Y_1^{(6)}\right)_{r,s}}\times\frac{\partial\left(Y_1^{(6)}\right)_{r,s}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

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## We have then

$$\frac{\partial H\left(\boldsymbol{W}\right)}{\partial\left(K_{11}^{(4)}\right)_{k,t}}=\frac{1}{2}\times\frac{\partial\left(y_{1}^{(7)}-t_{1}\right)^{2}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

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# Therefore

## We have

$$\frac{\partial z_i^{(7)}}{\partial \left(Y_1^{(6)}\right)_{r,s}} = w_{r,s}^{(7)}$$

## The final convolution is assuming that

$$\frac{\partial \left(Y_1^{(6)}\right)_{r,s}}{\partial \left(K_{11}^{(4)}\right)_{k,t}} = \frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial \left(K_{11}^{(4)}\right)_{k,t}}$$

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## We have

$$\frac{\partial z_i^{(7)}}{\partial \left(Y_1^{(6)}\right)_{r,s}} = w_{r,s}^{(7)}$$

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# Therefore

## We have

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(K_{11}^{(4)}\right)_{k,t}} = \frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}} \times \frac{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

## Then

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}} = f'\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]$$

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# Therefore

## We have

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(K_{11}^{(4)}\right)_{k,t}}=\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}\times\frac{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}{\partial\left(K_{11}^{(4)}\right)_{k,t}}$$

## Then

$$\frac{\partial f\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]}{\partial\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}=f'\left[\left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}\right]$$

# Finally, we have

## The equation

$$\frac{\partial \left(Y_1^{(4)}\right)_{2(r-1),2(s-1)}}{\partial \left(K_{11}^{(4)}\right)_{k,t}} = \left(Y^{(3)}\right)_{2(r-1)+k,2(s-1)+t}$$

# The Other Equations

## I will leave you to devise them

- They are a repetitive procedure.