Introduction to Machine Learning
Introduction to Support Vector Machines

Andres Mendez-Vazquez

June 13, 2018

Outline

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History

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Applications

Partial List

1. Predictive Control
   - Control of chaotic systems.
2. Inverse Geosounding Problem
   - It is used to understand the internal structure of our planet.
3. Environmental Sciences
   - Spatio-temporal environmental data analysis and modeling.
4. Protein Fold and Remote Homology Detection
   - In the recognition if two different species contain similar genes.
5. Facial expression classification
6. Texture Classification
7. E-Learning
8. Handwritten Recognition
9. AND counting....

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Separable Classes

Given

$$\boldsymbol{x}_i,\ i = 1, \cdots, N$$

A set of samples belonging to two classes ω_1, ω_2.

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$$\boldsymbol{x}_i,\ i = 1, \cdots, N$$

A set of samples belonging to two classes ω_1, ω_2.

Objective

We want to obtain a decision function as simple as

$$g(\boldsymbol{x}) = \boldsymbol{w}^T \boldsymbol{x} + w_0$$

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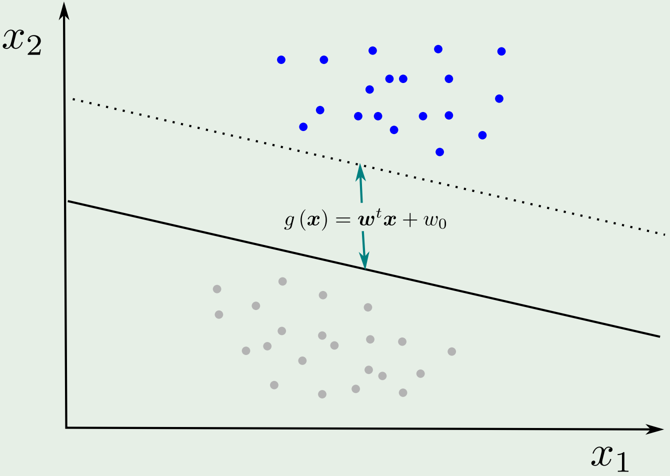
Such that we can do the following

A linear separation function $g(\boldsymbol{x}) = \boldsymbol{w}^t \boldsymbol{x} + w_0$



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In other words ...

We have the following samples

- For $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_m \in C_1$
- For $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n \in C_2$

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We want the following decision surfaces

- $\boldsymbol{w}^T \boldsymbol{x}_i + w_0 \geq 0$ for $d_i = +1$ if $\boldsymbol{x}_i \in C_1$
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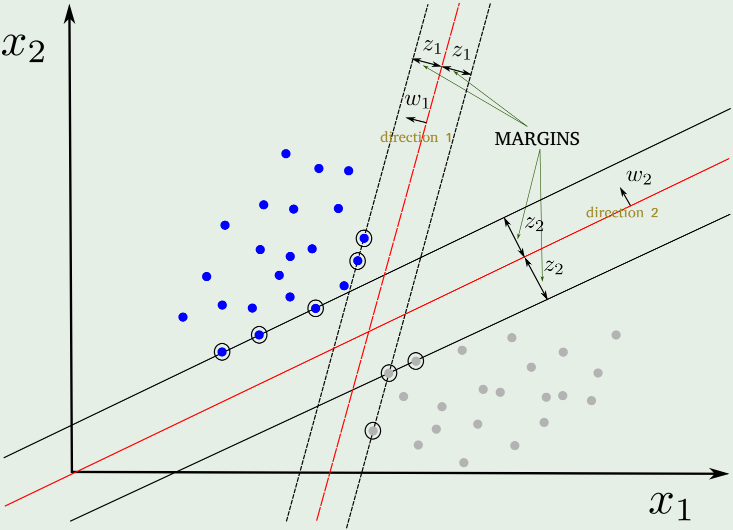
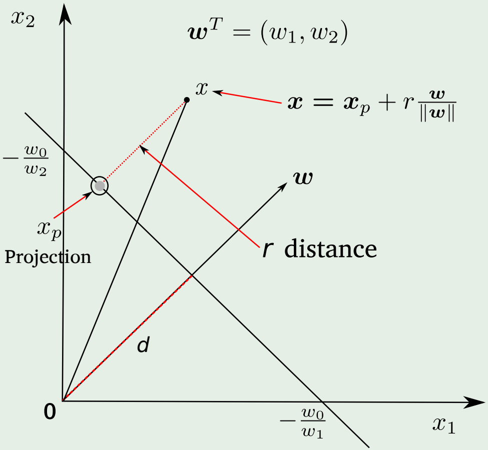
What do we want?

Our goal is to search for a direction w that gives the maximum possible margin



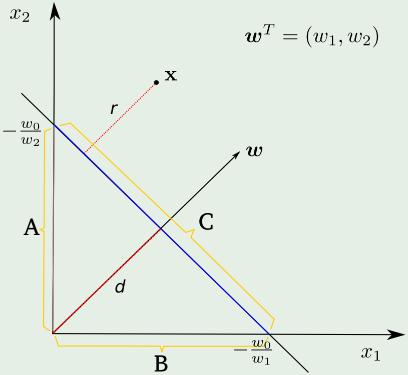
Remember

We have the following



A Little of Geometry

Thus



Then

$$d = \frac{|w_0|}{\sqrt{w_1^2 + w_2^2}}, \quad r = \frac{|g(\boldsymbol{x})|}{\sqrt{w_1^2 + w_2^2}} \tag{1}$$

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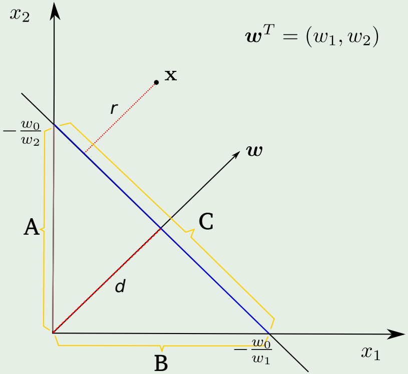

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First $d = \frac{|w_0|}{\sqrt{w_1^2 + w_2^2}}$

We can use the following rule in a triangle with a 90^o angle

$$Area = \frac{1}{2}Cd \quad (2)$$

In addition, the area can be calculated also as

$$Area = \frac{1}{2}AB \quad (3)$$

Thus

$$d = \frac{AB}{C}$$

Remark: Can you get the rest of values?

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What about $r = \frac{|g(\boldsymbol{x})|}{\sqrt{w_1^2 + w_2^2}}$?

First, remember

$$g\left(\boldsymbol{x}_p\right) = 0 \text{ and } \boldsymbol{x} = \boldsymbol{x}_p + r\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} \tag{4}$$

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$$\begin{aligned}
g\left(\boldsymbol{x}\right) =& \boldsymbol{w}^T\left[\boldsymbol{x}_p + r\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\right] + w_0 \\
=& \boldsymbol{w}^T\boldsymbol{x}_p + w_0 + r\frac{\boldsymbol{w}^T\boldsymbol{w}}{\|\boldsymbol{w}\|} \\
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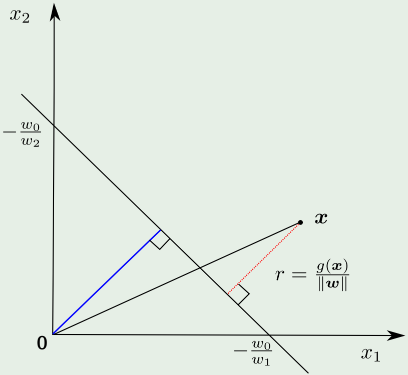
This has the following interpretation

The distance from the projection



Now

We know that the straight line that we are looking for looks like

$$\boldsymbol{w}^T x + w_0 = 0 \tag{5}$$

What about something like this

$$\boldsymbol{w}^T x + w_0 = \delta \tag{6}$$

Clearly

This will be above or below the initial line $\boldsymbol{w}^T x + w_0 = 0$.

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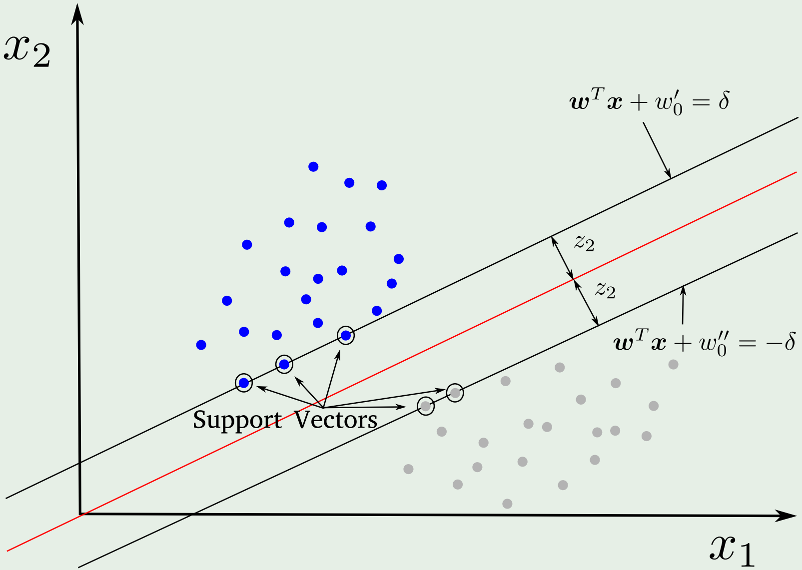
Come back to the hyperplanes

We have then for each border support line an specific bias!!!



Then, normalize by δ

The new margin functions

- $\boldsymbol{w}'^{T}\mathbf{x} + w_{10} = 1$
- $\boldsymbol{w}'^{T}\mathbf{x} + w_{01} = -1$

where $\boldsymbol{w}' = \frac{\boldsymbol{w}}{\delta}$, $w_{10} = \frac{w_0'}{\delta}$, and $w_{01} = \frac{w_0''}{\delta}$

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Now, we come back to the middle separator hyperplane, but with the normalized term

- $\boldsymbol{w}^T\mathbf{x}_i + w_0 \geq \boldsymbol{w}'^T\mathbf{x} + w_{10}$ for $d_i = +1$
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  - Where w_0 is the bias of that central hyperplane!! And the $\boldsymbol{w}$ is the normalized direction of $\boldsymbol{w}'$

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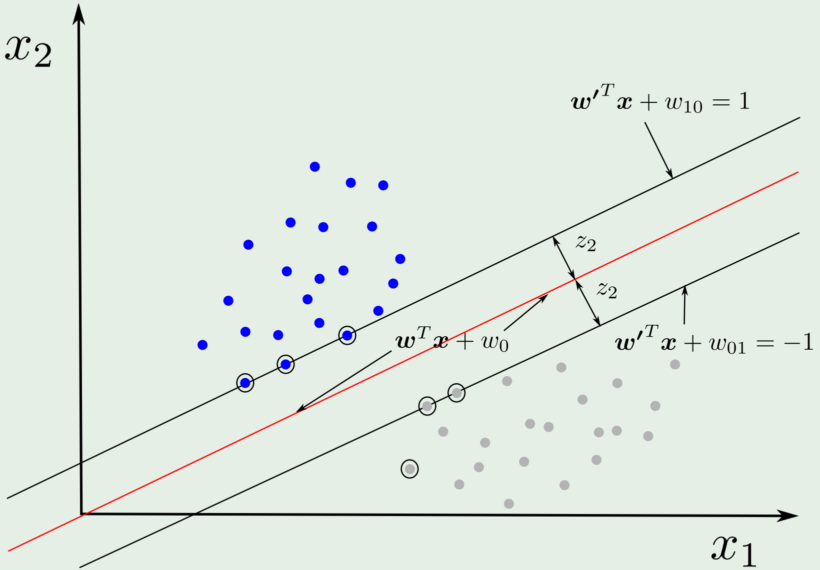
The meaning of what I am saying!!!



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A little about Support Vectors

They are the vectors (Here, we assume that w)

$\boldsymbol{x}_i$ such that $\boldsymbol{w}^T\boldsymbol{x}_i + w_0 = 1$ or $\boldsymbol{w}^T\boldsymbol{x}_i + w_0 = -1$

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Properties

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Now, we can resume the decision rule for the hyperplane

For the support vectors

$$g\left(\boldsymbol{x}_i\right) = \boldsymbol{w}^T \boldsymbol{x}_i + w_0 = -(+)1 \text{ for } d_i = -(+)1 \quad (7)$$

Implies

The distance to the support vectors is:

$$r = \frac{g\left(\boldsymbol{x}_i\right)}{\|\boldsymbol{w}\|} = \begin{cases} \frac{1}{\|\boldsymbol{w}\|} & \text{if } d_i = +1 \\ -\frac{1}{\|\boldsymbol{w}\|} & \text{if } d_i = -1 \end{cases}$$

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Therefore ...

We want the optimum value of the margin of separation as

$$\rho = \frac{1}{||\boldsymbol{w}||} + \frac{1}{||\boldsymbol{w}||} = \frac{2}{||\boldsymbol{w}||} \tag{8}$$

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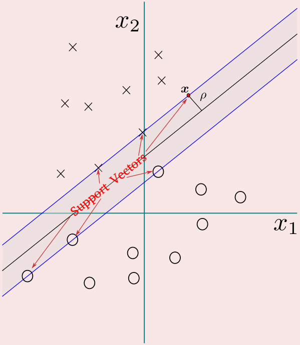

Outline

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Thus

If we want to maximize

$$\rho = \frac{2}{||\boldsymbol{w}||}$$

We instead to minimize

$$||w|| = \sqrt{w^T w}$$

Or to minimize, after all we only need the direction of the vector w

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Under the restrictions

Then, we have the samples with labels

$$T = \{(\boldsymbol{x}_i, d_i)\}_{i=1}^{N}$$

Then we can put the decision rule as

$$d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0\right) \geq 1 \; i = 1, \cdots, N$$

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Then, we have the optimization problem

The optimization problem

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s.t. $d_i(\boldsymbol{w}^T\boldsymbol{x}_i + w_0) \geq 1 \; i = 1, \cdots, N$

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- The cost functions $\Phi\left(\boldsymbol{w}\right)$ is convex.
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Outline

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Then, Rewriting The Optimization Problem

The optimization with equality constraints

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Then, for our problem

Using the Lagrange Multipliers (We will call them α_i)

We obtain the following cost function that we want to minimize

$$J(\boldsymbol{w}, w_0, \boldsymbol{\alpha}) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} - \sum_{i=1}^{N}\alpha_i[d_i(\boldsymbol{w}^T\mathbf{x}_i + w_0) - 1]$$

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Observation

- Minimize with respect to $\mathbf{w}$ and w_0.
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Outline

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Karush-Kuhn-Tucker Conditions

First An Inequality Constrained Problem P

$$
\begin{aligned}
\min \quad & f(\boldsymbol{x}) & \\
s.t \quad & g_1(\boldsymbol{x}) & = 0 \\
& \vdots & \\
& g_N(\boldsymbol{x}) & = 0
\end{aligned}
$$

A really minimal version!!! Hey, it is a patch work!!!

A point x is a local minimum of an equality constrained problem P only if a set of non-negative α_j's may be found such that:

$$\nabla L(x, \alpha) = \nabla f(x) - \sum_{i=1}^{N} \alpha_i \nabla g_i(x) = 0$$

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Karush-Kuhn-Tucker Conditions

Important

Think about this each constraint correspond to a sample in both classes, thus

- The corresponding α_i's are going to be zero after optimization, if a constraint is not active i.e. $d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0 \right) - 1 \neq 0$ (Remember Maximization).

Again the Support Vectors

This actually defines the idea of support vectors!!!

Thus

Only the α_i's with active constraints (Support Vectors) will be different from zero when $d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0 \right) - 1 = 0$.

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The necessary conditions for optimality

Condition 1

$$\frac{\partial J\left(\boldsymbol{w}, w_0, \boldsymbol{\alpha}\right)}{\partial \boldsymbol{w}} = 0$$

Condition 2

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Using the conditions

We have the first condition

$$\frac{\partial J(\boldsymbol{w}, w_0, \alpha)}{\partial \boldsymbol{w}} = \frac{\partial \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w}}{\partial \boldsymbol{w}} - \frac{\partial \sum_{i=1}^{N} \alpha_i [d_i(\boldsymbol{w}^T\boldsymbol{x}_i + w_0) - 1]}{\partial \boldsymbol{w}} = 0$$

$$\frac{\partial J(\boldsymbol{w}, w_0, \alpha)}{\partial \boldsymbol{w}} = \frac{1}{2}(\boldsymbol{w} + \boldsymbol{w}) - \sum_{i=1}^{N} \alpha_i d_i \boldsymbol{x}_i$$

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In a similar way ...

We have by the second optimality condition

$$\sum_{i=1}^{N} \alpha_i d_i = 0$$

Note

$$\alpha_i \left[d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0 \right) - 1 \right] = 0$$

Because the constraint vanishes in the optimal solution i.e. $\alpha_i = 0$ or $d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + w_0 \right) - 1 = 0$.

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Thus

We need something extra

Our classic trick of transforming a problem into another problem

In this case

We use the Primal-Dual Problem for Lagrangian

Where

We move from a minimization to a maximization!!!

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Outline

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Duality Theorem

First Property

If the Primal has an optimal solution ($\boldsymbol{w}*$ and $\boldsymbol{\alpha}*$), the dual too.

Thus

In order to $\boldsymbol{w}*$ and $\boldsymbol{\alpha}*$ to be optimal solutions for the primal and dual problem respectively, It is necessary and sufficient that $\boldsymbol{w}*$:

- It is a feasible solution for the primal problem and

$$\begin{aligned}\Phi(\boldsymbol{w}*) &= J(\boldsymbol{w}*, w_0*, \boldsymbol{\alpha}*) \\ &= \min_{\boldsymbol{w}} J(\boldsymbol{w}*, w_0*, \boldsymbol{\alpha}*)\end{aligned}$$

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Reformulate our Equations

We have then

$$J\left(\boldsymbol{w}, w_0, \boldsymbol{\alpha}\right) = \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} - \sum_{i=1}^{N}\alpha_i d_i \boldsymbol{w}^T\mathbf{x}_i - w_0\sum_{i=1}^{N}\alpha_i d_i + \sum_{i=1}^{N}\alpha_i$$

Now for our 2nd optimality condition

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We have finally for the 1st Optimality Condition:

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$$Q(\boldsymbol{\alpha}) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_j d_i d_j \boldsymbol{x}_j^T \boldsymbol{x}_i$$

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From here, we have the problem

This is the problem that we really solve

Given the training sample $\{(\mathbf{x}_i, d_i)\}_{i=1}^{N}$, find the Lagrange multipliers $\{\alpha_i\}_{i=1}^{N}$ that maximize the objective function

$$Q(\alpha) = \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j d_i d_j \boldsymbol{x}_j^T \boldsymbol{x}_i$$

subject to the constraints

$$\sum_{i=1}^{N} \alpha_i d_i = 0 \tag{11}$$

$$\alpha_i \geq 0 \text{ for } i = 1, \cdots, N \tag{12}$$

Note

In the Primal, we were trying to minimize the cost function, for this it is necessary to maximize α. That is the reason why we are maximizing $Q(\alpha)$.

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Solving for $\boldsymbol{\alpha}$

We can compute $\boldsymbol{w}^*$ once we get the optimal α_i^* by using (Eq. 10)

$$\boldsymbol{w}^* = \sum_{i=1}^{N} \alpha_i^* d_i \boldsymbol{x}_i$$

In addition, we can compute the optimal bias w_0^* using the optimal weight, $\boldsymbol{w}^*$

For this, we use the positive margin equation:

$$g\left(\boldsymbol{x}^{(s)}\right) = \boldsymbol{w}^T \boldsymbol{x}^{(s)} + w_0 = 1$$

corresponding to a positive support vector.

Then

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Outline

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What do we need?

Until now, we have only a maximal margin algorithm

- All this work fine when the classes are separable
- Problem, What when they are not separable?
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Outline

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Map to a higher Dimensional Space

Assume that exist a mapping

$$\boldsymbol{x} \in \mathbb{R}^l \to \boldsymbol{y} \in \mathbb{R}^k$$

Then, it is possible to define the following mapping

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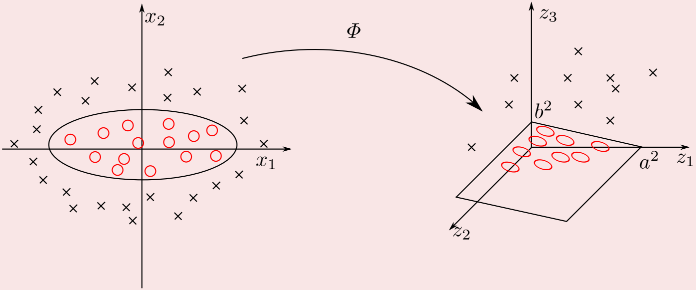

$$\Phi : (x_1, x_2) \rightarrow \left(x_1^2, \sqrt{2}x_1x_2, x_2^2\right)$$

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1 \rightarrow \frac{z_1}{a^2} + \frac{z_3}{b^2} = 1$$

Define a map to a higher Dimension

Nonlinear transformations

Given a series of nonlinear transformations

$$\{\phi_i(\boldsymbol{x})\}_{i=1}^{m}$$

from input space to the feature space.

We can define the decision surface as

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This allows us to define

The following vector

$$\phi(\boldsymbol{x}) = (\phi_0(\boldsymbol{x}), \phi_1(\boldsymbol{x}), \cdots, \phi_m(\boldsymbol{x}))^T$$

that represents the mapping.

From this mapping

We can define the following kernel function

$$K : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}$$

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Basic Idea

Something Notable

- The SVM uses the scalar product $\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle$ as a measure of similarity between $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, and of distance to the hyperplane.
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Finally

We define the decision surface as

$$\boldsymbol{w}^T \phi(\boldsymbol{x}) = 0 \quad (14)$$

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Thus

The term $\phi^T(\boldsymbol{x}_i)\,\phi(\boldsymbol{x})$

It represents the inner product of two vectors induced in the feature space induced by the input patterns.

We can introduce the inner-product kernel

$$K(\boldsymbol{x}_i, \boldsymbol{x}) = \phi^T(\boldsymbol{x}_i)\,\phi(\boldsymbol{x}) = \sum_{j=0}^{m} \phi_j(\boldsymbol{x}_i)\,\phi_j(\boldsymbol{x}) \quad (17)$$

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This allows to redefine the optimal hyperplane

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$$\sum_{i=1}^{N} \alpha_i d_i K\left(\boldsymbol{x}_i, \boldsymbol{x}\right) = 0 \quad (19)$$

Something Notable

Using kernels, we can avoid to go from:

$$\text{Input Space} \Longrightarrow \text{Mapping Space} \Longrightarrow \text{Inner Product} \quad (20)$$

By directly going from

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Important

Something Notable

The expansion of (Eq. 17) for the inner-product kernel $K(\boldsymbol{x}_i, \boldsymbol{x})$ is an important special case of that arises in functional analysis.

Mercer's Theorem

Mercer's Theorem

Let $K(\boldsymbol{x}, \boldsymbol{x}')$ be a continuous symmetric kernel that is defined in the closed interval $\boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}$ and likewise for $\boldsymbol{x}'$. The kernel $K(\boldsymbol{x}, \boldsymbol{x}')$ can be expanded in the series

$$K(\boldsymbol{x}, \boldsymbol{x}') = \sum_{i=1}^{\infty} \lambda_i \phi_i(\boldsymbol{x}) \phi_i(\mathbf{x}') \tag{22}$$

With

Positive coefficients, $\lambda_i > 0$ for all i.

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Mercer's Theorem

For this expression to be valid and or it to converge absolutely and uniformly

It is necessary and sufficient that the condition

$$\int_a^b \int_a^b K\left(\boldsymbol{x}, \boldsymbol{x'}\right) \psi\left(\boldsymbol{x}\right) \psi\left(\boldsymbol{x'}\right) d\boldsymbol{x} d\boldsymbol{x'} \geq 0 \tag{23}$$

holds for all ψ such that $\int_a^b \psi^2\left(\boldsymbol{x}\right) d\boldsymbol{x} < \infty$(Example of a quadratic norm for functions).

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Remarks

First

The functions $\phi_i(\boldsymbol{x})$ are called eigenfunctions of the expansion and the numbers λ_i are called eigenvalues.

Second

The fact that all of the eigenvalues are positive means that the kernel $K(x, x')$ is positive definite.

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Not only that

We have that

For $\lambda_i \neq 1$, the ith image of $\sqrt{\lambda_i}\phi_i(\boldsymbol{x})$ induced in the feature space by the input vector $\boldsymbol{x}$ is an eigenfunction of the expansion.

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The dimensionality of the feature space (i.e., the number of eigenvalues/ eigenfunctions) can be infinitely large.

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Outline

Example

Assume

$$\boldsymbol{x} \in \mathbb{R} \to \boldsymbol{y} = \begin{bmatrix} x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{bmatrix}$$

We can show that

$$\boldsymbol{y}_i^T \boldsymbol{y}_j = \left(\boldsymbol{x}_i^T \boldsymbol{x}_j\right)^2$$

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Example of Kernels

Polynomials

$$k(\boldsymbol{x}, \boldsymbol{z}) = (\boldsymbol{x}^T \boldsymbol{z} + 1)^q \; q > 0$$

Radial Basis Functions

$$k(\boldsymbol{x}, \boldsymbol{z}) = \exp\left(-\frac{\|\boldsymbol{x} - \boldsymbol{z}\|^2}{\sigma^2}\right)$$

Hyperbolic Tangents

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Now, How to select a Kernel?

We have a problem

Selecting a specific kernel and parameters is usually done in a try-and-see manner.

Thus

In general, the Radial Basis Functions kernel is a reasonable first choice.

Then

if this fails, we can try the other possible kernels.

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Thus, we have something like this

Step 1

Normalize the data.

Step 2

Use cross-validation to adjust the parameters of the selected kernel.

Step 3

Train against the entire dataset.

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Optimal Hyperplane for non-separable patterns

Important

We have been considering only problems where the classes are linearly separable.

Now

What happen when the patterns are not separable?

Thus, we can still build a separating hyperplane

But errors will happen in the classification... We need to minimize them...

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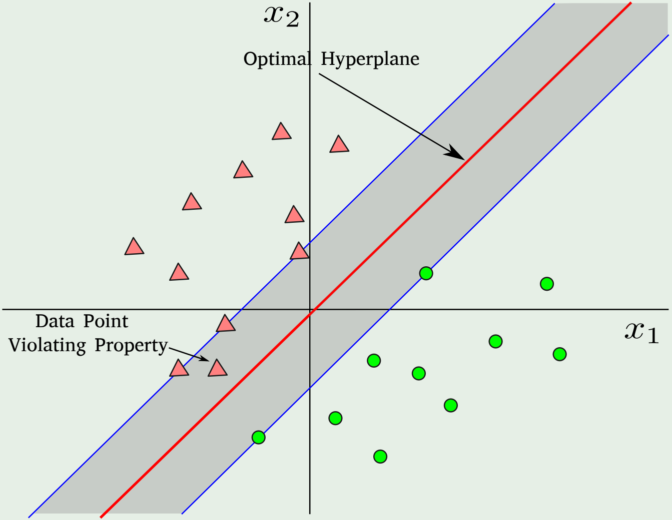
What if the following happens

Some data points invade the "margin" space



Fixing the Problem - Corinna's Style

The margin of separation between classes is said to be soft if a data point $(\boldsymbol{x}_i, d_i)$ violates the following condition

$$d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + b \right) \geq +1 \;\; i = 1, 2, ..., N \tag{24}$$

This violation can arise in one of two ways

The data point $(\boldsymbol{x}_i, d_i)$ falls inside the region of separation but on the right side of the decision surface - still correct classification.

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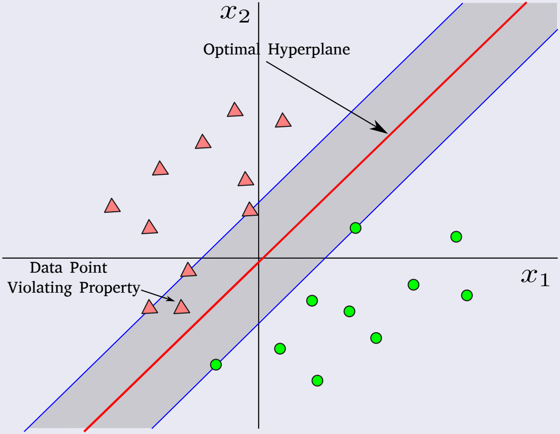

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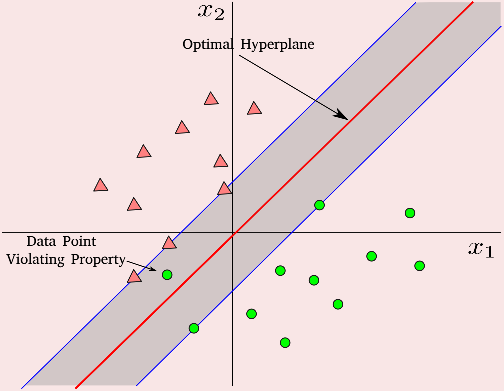

Solving the problem

What to do?

- We introduce a set of nonnegative scalar values $\{\xi_i\}_{i=1}^{N}$.

Introduce this into the decision rule

$$d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + b \right) \geq 1 - \xi_i \; i = 1, 2, ..., N \tag{25}$$

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The ξ_i are called slack variables

What?

In 1995, Corinna Cortes and Vladimir N. Vapnik suggested a modified maximum margin idea that allows for mislabeled examples.

Ok!!!

Instead of expecting to have constant margin for all the samples, the margin can change depending of the sample.

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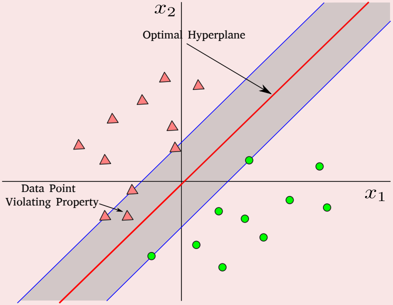

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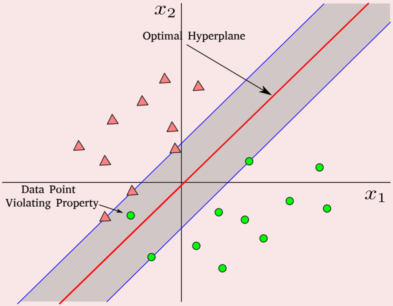

Support Vectors

We want

- Support vectors that satisfy equation (Eq. 25) even when $\xi_i > 0$

$$d_i \left(\boldsymbol{w}^T \boldsymbol{x}_i + b \right) \geq 1 - \xi_i \;\; i = 1, 2, ..., N$$

We want the following

We want to find an hyperplane

Such that average error is misclassified over all the samples

$$\frac{1}{N}\sum_{i=1}^{N}\mathrm{e}^{2} \tag{26}$$

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First Attempt Into Minimization

We can try the following

Given

$$I(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases} \quad (27)$$

Minimize the following

$$\Phi(\xi) = \sum_{i=1}^{N} I(\xi_i - 1) \quad (28)$$

with respect to the weight vector w subject to

1. $d_i \left(w^T x_i + b \right) \geq 1 - \xi_i \; i = 1, 2, ..., N$
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Minimization of $\Phi(\boldsymbol{\xi})$ with respect to $\mathbf{w}$ is a non-convex optimization problem that is NP-complete.

Thus, we need to use an approximation, maybe

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Now, we simplify the computations by integrating the vector w

$$\Phi(w, \xi) = \frac{1}{2} w^T w + C \sum_{i=1}^{N} \xi_i \tag{30}$$

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Minimizing the first term in (Eq. 30) is related to minimize the Vapnik–Chervonenkis dimension.

- Which is a measure of the capacity (complexity, expressive power, richness, or flexibility) of a statistical classification algorithm.

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The parameter C has to be selected by the user.

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Primal Problem

Problem, given samples $\{(\boldsymbol{x}_i, d_i)\}_{i=1}^{N}$

$$\min_{\boldsymbol{w},\boldsymbol{\xi}} \Phi\left(\boldsymbol{w}, \boldsymbol{\xi}\right) = \min_{\mathbf{w},\boldsymbol{\xi}} \left\{ \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} + C\sum_{i=1}^{N}\xi_i \right\}$$

$$\text{s.t. } d_i(\boldsymbol{w}^T\boldsymbol{x}_i + w_0) \geq 1 - \xi_i \text{ for } i = 1, \cdots, N$$

$$\xi_i \geq 0 \text{ for all } i$$

With C a user-specified positive parameter.

Outline

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Final Setup

Using Lagrange Multipliers and dual-primal method is possible to obtain the following setup

Given the training sample $\{(\mathbf{x}_i, d_i)\}_{i=1}^{N}$, find the Lagrange multipliers $\{\alpha_i\}_{i=1}^{N}$ that maximize the objective function

$$\min_{\alpha} Q(\alpha) = \min_{\alpha} \left\{ \sum_{i=1}^{N} \alpha_i - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j d_i d_j \boldsymbol{x}_j^T \boldsymbol{x}_i \right\}$$

subject to the constraints

$$\sum_{i=1}^{N} \alpha_i d_i = 0 \quad (31)$$

$$0 \leq \alpha_i \leq C \text{ for } i = 1, \cdots, N \quad (32)$$

where C is a user-specified positive parameter.

Remarks

Something Notable

- Note that neither the slack variables nor their Lagrange multipliers appear in the dual problem.
- The dual problem for the case of non-separable patterns is thus similar to that for the simple case of linearly separable patterns

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$$\boldsymbol{w}^* = \sum_{i=1}^{N_s} \alpha_i^* d_i \boldsymbol{x}_i$$

Where N_s is the number of support vectors.

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At saddle point, the derivative of the Lagrangian function for the primal problem:

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$$\alpha_i + \mu_i = C \tag{35}$$

Thus, we get if $\alpha_i < C$

Then $\mu_i > 0 \Rightarrow \xi_i = 0$

We may determine w_0

Using any data point (x_i, d_i) in the training set such that $0 \leq \alpha_i^* \leq C$. Then, given $\xi_i = 0$,

$$w_0^* = \frac{1}{d_i} - (w^*)^T x_i \tag{36}$$

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Nevertheless

It is better

To take the mean value of w_0^* from all such data points in the training sample (Burges, 1998).

- BTW He has a great book in SVM's "An Introduction to Support Vector Machines and Other Kernel-based Learning Methods"

Cinvestav