Introduction to Machine Learning Feature Generation

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Outline



- Introduction
- The Rotation Idea
- Solution
 - Scatter measure
- The Cost Function

2 Principal Components and Singular Value Decomposition

Introduction

Principal Component Analysis AKA Karhunen-Loeve Transform

- Projecting the Data
- Lagrange Multipliers
- The Process
- Example
- Singular Value Decomposition
 - Introduction
 - Building Such Solution
 - Image Compression

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1 Fisher Linear Discriminant

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What do we want?

What

• Given a set of measurements, the goal is to discover compact and informative representations of the obtained data.

Our Approach

 We want to "squeeze" in a relatively small number of features, leading to a reduction of the necessary feature space dimension.

Properties

Thus removing information redundancies - Usually produced and the measurement.

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What Methods we will see?

Fisher Linear Discriminant

- Squeezing to the maximum.
- From Many to One Dimension

Principal Component Analysi

- Not so much squeezing
- 2 You are willing to lose some information

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Rotation

Projecting

Projecting well-separated samples onto an arbitrary line usually produces a confused mixture of samples from all of the classes and thus produces poor recognition performance.

Something Notable

However, moving and rotating the line around might result in an orientation for which the projected samples are well separated.

Fisher linear discriminant (FLD)

It is a discriminant analysis seeking directions that are efficient for discriminating binary classification problem.

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Example

Example - From Left to Right the Improvement



This is actually comming from...

Classifier as

A machine for dimensionality reduction.

Initial Setup

We have:

• N d-dimensional samples $x_1, x_2, ..., x_N$

• N_i is the number of samples in class C_i for i=1,2.

Then, we ask for the projection of each x_i into the line by means of

$$y_i = \boldsymbol{w}^T \boldsymbol{x}_i$$

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Use the mean of each Class

Then

Select \boldsymbol{w} such that class separation is maximized

We then define the mean sample for ecah class

• $C_1 \Rightarrow m_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} x_i$ • $C_2 \Rightarrow m_2 = \frac{1}{N_2} \sum_{i=1}^{N_2} x_i$

Ok!!! This is giving us a measure of distance

Thus, we want to maximize the distance the projected means:

$$m_1 - m_2 = \boldsymbol{w}^T \left(\boldsymbol{m}_1 - \boldsymbol{m}_2 \right) \tag{2}$$

where $m_k = \boldsymbol{w}^T \boldsymbol{m}_k$ for k = 1, 2.

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However

We could simply seek

$$\max \boldsymbol{w}^{T} \left(\boldsymbol{m}_{1} - \boldsymbol{m}_{2} \right)$$
$$s.t. \sum_{i=1}^{d} w_{i} = 1$$

After all

We do not care about the magnitude of $oldsymbol{w}.$

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Example

Here, we have the problem



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Fixing the Problem

To obtain good separation of the projected data

The difference between the means should be large relative to some measure of the standard deviations for each class.

We define a SCATTER measure (Based in the Sample Variance)

$$s_k^2 = \sum_{x_i \in C_k} \left(w^T x_i - m_k \right)^2 = \sum_{y_i = w^T x_i \in C_k} (y_i - m_k)^2$$
(3)

We define then within-class variance for the whole dataa

$$s_1^2 + s_2^2$$

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Finally, a Cost Function

The between-class variance

$$(m_1 - m_2)^2$$

The Fisher criterion

between-class variance within-class variance

Finally

$$J(\boldsymbol{w}) = rac{(m_1 - m_2)^2}{s_1^2 + s_2^2}$$

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 $J(w) = \frac{(m_1 - m_2)^2}{s_1^2 + s_2^2}$

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We use a transformation to simplify our life



$$J(w) = \frac{\left(w^{T}m_{1} - w^{T}m_{2}\right)^{2}}{\sum_{y_{i} = w^{T}w_{i} \in C_{1}} (y_{i} - m_{k})^{2} + \sum_{y_{i} = w^{T}w_{i} \in C_{2}} (y_{i} - m_{k})^{2}}$$

Second

$$=\frac{\left(\boldsymbol{w}^{T}\boldsymbol{m}_{1}-\boldsymbol{w}^{T}\boldsymbol{m}_{2}\right)\left(\boldsymbol{w}^{T}\boldsymbol{m}_{1}-\boldsymbol{w}^{T}\boldsymbol{m}_{2}\right)^{T}}{\sum_{y_{i}=\boldsymbol{w}^{T}\boldsymbol{x}_{1}\in C_{1}}\left(\boldsymbol{w}^{T}\boldsymbol{x}_{i}-\boldsymbol{m}_{k}\right)\left(\boldsymbol{w}^{T}\boldsymbol{x}_{i}-\boldsymbol{m}_{k}\right)^{T}+\sum_{y_{i}=\boldsymbol{w}^{T}\boldsymbol{x}_{i}\in C_{2}}\left(\boldsymbol{w}^{T}\boldsymbol{x}_{i}-\boldsymbol{m}_{k}\right)\left(\boldsymbol{w}^{T}\boldsymbol{x}_{i}-\boldsymbol{m}_{k}\right)^{T}}$$

Third

$$=\frac{w^{T}\left(m_{1}-m_{2}\right)\left(w^{T}\left(m_{1}-m_{2}\right)\right)^{T}}{\sum_{y_{i}=w^{T}x_{i}\in C_{1}}w^{T}\left(x_{i}-m_{1}\right)\left(w^{T}\left(x_{i}-m_{1}\right)\right)^{T}+\sum_{y_{i}=w^{T}x_{i}\in C_{2}}w^{T}\left(x_{i}-m_{2}\right)\left(w^{T}\left(x_{i}-m_{2}\right)\right)^{T}}$$

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Transformation

Fourth

$$= \frac{w^T (m_1 - m_2) (m_1 - m_2)^T w}{\sum_{y_i = w^T x_i \in C_1} w^T (x_i - m_1) (x_i - m_1)^T w + \sum_{y_i = w^T x_i \in C_2} w^T (x_i - m_2) (x_i - m_2)^T w}$$

Fifth

$$w^{T}\left(m_{1}-m_{2}
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$$\sum_{y_1=w^T x_1 \in C_1} (x_i - m_1) (x_1 - m_1)^T + \sum_{y_1=w^T x_1 \in C_2} (x_i - m_2) (x_1 - m_2)^T \right] w_1 = 0$$

Now Kename

$$J(w) = \frac{w^T S_B w}{w^T S_w w} \tag{8}$$

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$$= \frac{w^T (m_1 - m_2) (m_1 - m_2)^T w}{w^T \left[\sum_{y_i = w^T x_i \in C_1} (x_i - m_1) (x_i - m_1)^T + \sum_{y_i = w^T x_i \in C_2} (x_i - m_2) (x_i - m_2)^T \right] w}$$

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(8)

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Thus

$$\frac{dJ(\boldsymbol{w})}{d\boldsymbol{w}} = \frac{d\left(\boldsymbol{w}^{T}\boldsymbol{S}_{B}\boldsymbol{w}\right)\left(\boldsymbol{w}^{T}\boldsymbol{S}_{w}\boldsymbol{w}\right)^{-1}}{d\boldsymbol{w}} = 0$$
(9)

Then

$$\frac{dJ(\boldsymbol{w})}{d\boldsymbol{w}} = \left(\boldsymbol{S}_{B}\boldsymbol{w} + \boldsymbol{S}_{B}^{T}\boldsymbol{w}\right)\left(\boldsymbol{w}^{T}\boldsymbol{S}_{w}\boldsymbol{w}\right)^{-1} - \left(\boldsymbol{w}^{T}\boldsymbol{S}_{B}\boldsymbol{w}\right)\left(\boldsymbol{w}^{T}\boldsymbol{S}_{w}\boldsymbol{w}\right)^{-2}\left(\boldsymbol{S}_{w}\boldsymbol{w} + \boldsymbol{S}_{w}^{T}\boldsymbol{w}\right) = 0$$
(10)

Now because the symmetry in ${m S}_B$ and ${m S}_R$

$$\frac{dJ(\boldsymbol{w})}{d\boldsymbol{w}} = \frac{\boldsymbol{S}_B}{(\boldsymbol{w}^T \boldsymbol{S}_{\boldsymbol{w}} \boldsymbol{w})} - \frac{\boldsymbol{w}^T \boldsymbol{S}_B \boldsymbol{w} \boldsymbol{S}_{\boldsymbol{w}} \boldsymbol{w}}{(\boldsymbol{w}^T \boldsymbol{S}_{\boldsymbol{w}} \boldsymbol{w})^2} = 0$$
(11)

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Then

$$\frac{dJ(\boldsymbol{w})}{d\boldsymbol{w}} = \left(\boldsymbol{S}_{B}\boldsymbol{w} + \boldsymbol{S}_{B}^{T}\boldsymbol{w}\right)\left(\boldsymbol{w}^{T}\boldsymbol{S}_{w}\boldsymbol{w}\right)^{-1} - \left(\boldsymbol{w}^{T}\boldsymbol{S}_{B}\boldsymbol{w}\right)\left(\boldsymbol{w}^{T}\boldsymbol{S}_{w}\boldsymbol{w}\right)^{-2}\left(\boldsymbol{S}_{w}\boldsymbol{w} + \boldsymbol{S}_{w}^{T}\boldsymbol{w}\right) = 0$$
(10)

Now because the symmetry in $oldsymbol{S}_B$ and $oldsymbol{S}_w$

$$\frac{dJ(\boldsymbol{w})}{d\boldsymbol{w}} = \frac{\boldsymbol{S}_B}{\left(\boldsymbol{w}^T \boldsymbol{S}_w \boldsymbol{w}\right)} - \frac{\boldsymbol{w}^T \boldsymbol{S}_B \boldsymbol{w} \boldsymbol{S}_w \boldsymbol{w}}{\left(\boldsymbol{w}^T \boldsymbol{S}_w \boldsymbol{w}\right)^2} = 0$$
(11)

Thus

$$\frac{dJ(\boldsymbol{w})}{d\boldsymbol{w}} = \frac{\boldsymbol{S}_B}{\left(\boldsymbol{w}^T \boldsymbol{S}_w \boldsymbol{w}\right)} - \frac{\boldsymbol{w}^T \boldsymbol{S}_B \boldsymbol{w} \boldsymbol{S}_w \boldsymbol{w}}{\left(\boldsymbol{w}^T \boldsymbol{S}_w \boldsymbol{w}\right)^2} = 0$$
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Then

$$(\boldsymbol{w}^T \boldsymbol{S}_w \boldsymbol{w}) \boldsymbol{S}_B \boldsymbol{w} = (\boldsymbol{w}^T \boldsymbol{S}_B \boldsymbol{w}) \boldsymbol{S}_w \boldsymbol{w}$$
 (13)

First

$$S_B w = (m_1 - m_2) (m_1 - m_2)^T w = \alpha (m_1 - m_2)$$
 (14)

Where $lpha = (oldsymbol{m}_1 - oldsymbol{m}_2)^* oldsymbol{w}$ is a simple constant

It means that $oldsymbol{S}_Boldsymbol{w}$ is always in the direction $oldsymbol{m}_1-oldsymbol{m}_2!!!$

In addition

 $oldsymbol{w}^Toldsymbol{S}_woldsymbol{w}$ and $oldsymbol{w}^Toldsymbol{S}_Boldsymbol{w}$ are constants

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Finally

$$\boldsymbol{S}_{w} \boldsymbol{w} \propto (\boldsymbol{m}_{1} - \boldsymbol{m}_{2}) \Rightarrow \boldsymbol{w} \propto \boldsymbol{S}_{w}^{-1} (\boldsymbol{m}_{1} - \boldsymbol{m}_{2})$$
 (15)



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Once the data is transformed into y_i

• Use a threshold $y_0 \Rightarrow x \in C_1$ iff $y(x) \ge y_0$ or $x \in C_2$ iff $y(x) < y_0$

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- Use a threshold $y_0 \Rightarrow x \in C_1$ iff $y(x) \ge y_0$ or $x \in C_2$ iff $y(x) < y_0$
- Or ML with a Gussian can be used to classify the new transformed data using a Naive Bayes (Central Limit Theorem and $y = w^T x$ sum of random variables).

Your Reading Material, it is about the Multiclass

4.1.6 Fisher's discriminant for multiple classes AT "Pattern Recognition" by Bishop

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Fisher Linear Discriminant

- Introduction
- The Rotation Idea
- Solution
 - Scatter measure
- The Cost Function

2 Principal Components and Singular Value Decomposition

Introduction

- Principal Component Analysis AKA Karhunen-Loeve Transform
 - Projecting the Data
 - Lagrange Multipliers
 - The Process
 - Example
- Singular Value Decomposition
 - Introduction
 - Building Such Solution
 - Image Compression

Did you noticed?

That Rotations really do not exist

• Actually, they are mappings or projections in linear algebra

Thus, Can we get more powerful mappings

• To obtain better features

Clearly... Yes

 For example, Principal Components or Singular Value Decomposition's

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Also Known as Karhunen-Loeve Transform

Setup

• Consider a data set of observations $\{ {m x}_n \}$ with n=1,2,...,N and ${m x}_{m n} \in R^d.$

Goal

Project data onto space with dimensionality $m < d \ensuremath{\left(\mathrm{We \ assume \ } m \ensuremath{m} m \ensuremath{\left(\mathrm{We \ assume \ } m \ensuremath{\left(\mathrm{We \ assume \ } m \ensuremath{\left(\mathrm{We \ assume \ } m \ensuremath{m} m \ensurem$

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Goal

Project data onto space with dimensionality m < d (We assume m is given)

Dimensional Variance

Remember the Variance Sample in $\ensuremath{\mathbb{R}}$

$$VAR(X) = \frac{\sum_{i=1}^{N} (x_i - \overline{x})^2}{N - 1}$$
(16)

You can do the same in the case of two variables X and Y $COV(x,y) = \frac{\sum_{i=1}^{N} (x_i - \overline{x}) (y_i - \overline{y})}{N-1}$ (17)

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Now, Define

Given the data

$$oldsymbol{x}_1, oldsymbol{x}_2, ..., oldsymbol{x}_N$$

(18)

where \boldsymbol{x}_i is a column vector

Construct the sample mean

$$\overline{x} = rac{1}{N}\sum_{i=1}^N x_i$$

Center data

$$oldsymbol{x}_1-\overline{oldsymbol{x}},oldsymbol{x}_2-\overline{oldsymbol{x}},...,oldsymbol{x}_N-\overline{oldsymbol{x}}$$
 (20

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 (20)

(18)

(19)

Build the Sample Mean

The Covariance Matrix

$$S = \frac{1}{N-1} \sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^T$$
(21)

- lacksim lacksim The ijth value of S is equivalent to σ^2_{ij}
- igleon The iith value of S is equivalent to $\sigma^2_{ii}.$

Build the Sample Mean

The Covariance Matrix

$$S = \frac{1}{N-1} \sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^T$$
(21)

Properties

- The *ij*th value of S is equivalent to σ_{ij}^2 .
- 2 The *ii*th value of S is equivalent to σ_{ii}^2 .

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Using S to Project Data

For this we use a u_1

• with $oldsymbol{u}_1^Toldsymbol{u}_1=1,$ an orthonormal vector

Question

What is the Sample Variance of the Projected Data?

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Thus we have

Variance of the projected data

$$\frac{1}{N-1}\sum_{i=1}^{N} \left[\boldsymbol{u}_{1}\boldsymbol{x}_{i} - \boldsymbol{u}_{1}\overline{\boldsymbol{x}}\right] = \boldsymbol{u}_{1}^{T}S\boldsymbol{u}_{1}$$
(22)

Use Lagrange Multipliers to Maximize

$$\boldsymbol{u}_1^T \boldsymbol{S} \boldsymbol{u}_1 + \lambda_1 \left(1 - \boldsymbol{u}_1^T \boldsymbol{u}_1 \right) \tag{23}$$

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Use Lagrange Multipliers to Maximize

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Derive by \boldsymbol{u}_1



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Derive by \boldsymbol{u}_1

$S \boldsymbol{u}_1 = \lambda_1 \boldsymbol{u}_1$

Then

We get

 \boldsymbol{u}_1 is an eigenvector of S.

If we left-multiply by $m{u}_1$

$$oldsymbol{u}_1^T oldsymbol{S} oldsymbol{u}_1 = \lambda_1$$

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(24)

Derive by \boldsymbol{u}_1

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What about the second eigenvector \boldsymbol{u}_2

We have the following optimization problem

$$\begin{array}{l} \max \ \boldsymbol{u}_2^T S \boldsymbol{u}_2 \\ \text{s.t.} \ \boldsymbol{u}_2^T \boldsymbol{u}_2 = 1 \\ \boldsymbol{u}_2^T \boldsymbol{u}_1 = 0 \end{array}$$

Lagrangian

 $L\left(oldsymbol{u}_{2},\lambda_{1},\lambda_{2}
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Explanation

First the constrained minimization

• We want to to maximize $oldsymbol{u}_2^T S oldsymbol{u}_2$

Given that the second eigenvector is orthonormal

ullet We have then $oldsymbol{u}_2^Toldsymbol{u}_2=1$

Under orthonormal vectors

• The covariance goes to zero $cov (\boldsymbol{u}_1, \boldsymbol{u}_2) = \boldsymbol{u}_2^T S \boldsymbol{u}_1 = \boldsymbol{u}_2 \lambda_1 \boldsymbol{u}_1 = \lambda_1 \boldsymbol{u}_1^T \boldsymbol{u}_2 = 0$

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Meaning

The PCA's are perpendicular

$$L(\boldsymbol{u}_2, \lambda_1, \lambda_2) = \boldsymbol{u}_2^T S \boldsymbol{u}_2 - \lambda_1 \left(\boldsymbol{u}_2^T \boldsymbol{u}_2 - 1 \right) - \lambda_2 \left(\boldsymbol{u}_2^T \boldsymbol{u}_1 - 0 \right)$$

The the derivative with respect to $oldsymbol{u}_2$

$$\frac{\partial L\left(\boldsymbol{u}_{2},\lambda_{1},\lambda_{2}\right)}{\partial \boldsymbol{u}_{2}}=S\boldsymbol{u}_{2}-\lambda_{1}\boldsymbol{u}_{2}-\lambda_{2}\boldsymbol{u}_{1}=0$$

Then, we left multiply $oldsymbol{u}_1$

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Then, we have that

We have because of Orthogonality

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Implying the classic solution

• $oldsymbol{u}_2$ is the eigenvector of S with second largest eigenvalue $\lambda_2.$

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Implying the classic solution

• u_2 is the eigenvector of S with second largest eigenvalue λ_2 .

Thus

Variance will be the maximum when

$$\boldsymbol{u}_1^T S \boldsymbol{u}_1 = \lambda_1 \tag{26}$$

is set to the largest eigenvalue. Also know as the First Principal Component

By Induction

It is possible for *M*-dimensional space to define *M* eigenvectors $u_1, u_2, ..., u_M$ of the data covariance S corresponding to $\lambda_1, \lambda_2, ..., \lambda_M$ that maximize the variance of the projected data.

Computational Cost of PCA

- Full eigenvector decomposition $O\left(d^3\right)$
- $igodoldsymbol{igo$
- Use the Expectation Maximization Algorithm

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We have the following steps

Determine covariance matrix

$$S = \frac{1}{N-1} \sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^T$$
(27)

Generate the decomposition

 $S = U\Sigma U^T$

With

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• Eigenvalues in Σ and eigenvectors in the columns of U.

Project samples x_i into subspaces dim=k

$$z_i = U_K^T \boldsymbol{x}_i$$

• With U_k is a matrix with k columns

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Example

From Bishop



Example



Example

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What happened with no-square matrices

We can still diagonalize it

Thus, we can obtain certain properties.

We want to avoid the problems with

 $S^{-1}AS$

The eigenvectors in S have three big problems

- They are usually not orthogonal...
- There are not always enough eigenvectors.
- $A \boldsymbol{x} = \lambda \boldsymbol{x}$ requires A to be square.

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- $Ax = \lambda x$ requires A to be square.

Therefore, we can look at the following problem

We have a series of vectors

 $\{ {m x}_1, {m x}_2, ..., {m x}_d \}$

Then imagine a set of projection vectors and differences

 $\{oldsymbol{eta}_1,oldsymbol{eta}_2,...,oldsymbol{eta}_d\}$ and $\{oldsymbol{lpha}_1,oldsymbol{lpha}_2,...,oldsymbol{lpha}_d\}$

We want to know a little bit of the relations between them

 After all, we are looking at the possibility of using them for our problem Therefore, we can look at the following problem

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Using the Hypotenuse

A little bit of Geometry, we get



Therefore

We have two possible quantities for each j

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Then, we can minimize and maximize given that $x_i^{*}x_j^{*}$ is a constant t

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Actually this is know as the dual problem (Weak Duality)

An example of this

 $\begin{array}{l} \min \ \boldsymbol{w}^T \boldsymbol{x} \\ s.t \mathsf{A} \boldsymbol{x} \leq \boldsymbol{b} \\ \boldsymbol{x} \geq 0 \end{array}$

Then, using what is know as slack variabless

 $A\boldsymbol{x} + A'\boldsymbol{x} = b$

Each row lives in the column space, but the y, lives in the column space.

 $(Aoldsymbol{x}+A'oldsymbol{x})_i
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Example of such Slack Matrix

$$\left(Aoldsymbol{x}+A'oldsymbol{x}
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ight]oldsymbol{x}+\left[egin{array}{ccc} 0&0\\0&-1\\-1&0\end{array}
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Element in the column space of dimensionality have three dimensions

But in the row space their dimension is 2

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Stack such vectors that in the *d*-dimensional space the

$\bullet\,$ In a matrix A of $n\times d$

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Why? Do you remember the Projection to a single vector p?

Definition of the projection under unitary vector

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It is possible to ask to maximize the longitude of such vector (Singular Vector)

$$oldsymbol{v}_1 = rg\max_{\|oldsymbol{v}\|=1} \|Aoldsymbol{v}\|$$

Then, we can define the following singular values

 $\sigma_1\left(A\right) = \|A\boldsymbol{v}_1\|$



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- The **best-fit line problem** describes the problem of finding the best line for a set of data points, where the quality of the line is measured by the sum of squared (perpendicular) distances of the points to the line.
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Generalization

 This can be transferred to higher dimensions: One can find the best-fit d-dimensional subspace, so the subspace which minimizes the sum of the squared distances of the points to the subspace

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Then, in a Greedy Fashion

The second singular vector $oldsymbol{v}_2$

$$oldsymbol{v}_2 = rg\max_{oldsymbol{v}ot oldsymbol{v}_1, \|oldsymbol{v}\|=1} \|Aoldsymbol{v}\|$$

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Stop when we have found all the following vectors:

 $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_r$

As singular vectors and

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Proving that the strategy is good

Theorem

• Let A be an $n \times d$ matrix where $v_1, v_2, ..., v_r$ are the singular vectors defined above. For $1 \le k \le r$, let V_k be the subspace spanned by $v_1, v_2, ..., v_k$. Then for each k, V_k is the best-fit k-dimensional subspace for A.

Proof

For k = 1

• What about k = 2? Let W be a best-fit 2- dimensional subspace for A.

For any basis $oldsymbol{w}_1,oldsymbol{w}_2$ of W

|Aw₁|² + |Aw₂|² is the sum of the squared lengths of the projections of the rows of A to W.

Now, choose a basis w_1, w_2 so that w_2 is perpendicular to v_1

• This can be a unit vector perpendicular to v_1 projection in W_2

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In a similar way for k

V_k is at least as good as W and hence is optimal.

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Remarks

Every Matrix has a singular value decomposition

 $A = U \Sigma V^T$

Where

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- The columns of V are an orthonormal basis for the row space.
- The Σ is diagonal and the entries on its diagonal $\sigma_i = \Sigma_{ii}$ are positive real numbers, called the singular values of A.

Properties of the Singular Value Decomposition

First

The eigenvalues of the symmetric matrix ${\cal A}^T{\cal A}$ are equal to the square of the singular values of ${\cal A}$

$$A^TA = V\Sigma U^T U^T \Sigma V^T = V\Sigma^2 V^T$$

Second

The rank of a matrix is equal to the number of non-zero singular values.

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Singular Value Decomposition as Sums

The singular value decomposition can be viewed as a sum of rank 1 matrices

$$A = A_1 + A_2 + \ldots + A_R$$



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Singular Value Decomposition as Sums

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Why?

$$\boldsymbol{u}_{1}\boldsymbol{A} = \boldsymbol{U} \begin{pmatrix} \sigma_{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_{2} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \sigma_{R} \end{pmatrix} \boldsymbol{V}^{T} = \begin{pmatrix} \boldsymbol{u}_{1} & \boldsymbol{u}_{2} & \cdots & \boldsymbol{u}_{R} \end{pmatrix} \begin{pmatrix} \sigma_{1}\boldsymbol{v}_{1}^{T} \\ \sigma_{2}\boldsymbol{v}_{2}^{T} \\ \vdots \\ \vdots \\ \sigma_{R}\boldsymbol{v}_{R}^{T} \end{pmatrix}$$
$$= \sigma_{1}\boldsymbol{u}_{1}\boldsymbol{v}_{1}^{T} + \sigma_{2}\boldsymbol{u}_{2}\boldsymbol{v}_{2}^{T} + \cdots + \sigma_{R}\boldsymbol{u}_{R}\boldsymbol{v}_{R}^{T}$$

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Truncating the singular value decomposition allows us to represent the matrix with less parameters



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For a 512×512

- Full Representation $512 \times 512 = 262, 144$
- Rank 10 approximation $512 \times 10 + 10 + 10 \times 512 = 10,250$

• Rank 40 approximation $512 \times 40 + 40 + 40 \times 512 = 41,000$

• Rank 80 approximation $512 \times 80 + 80 + 80 \times 512 = 82,000$

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