Introduction to Machine Learning

Introduction to Bayesian Classification

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June 3, 2020

Outline

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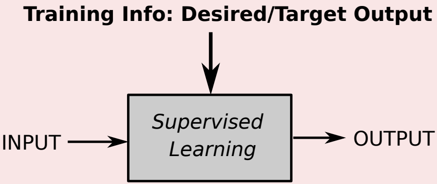
Classification Problem

Goal

Given $\boldsymbol{x}_{new}$, provide $f(\boldsymbol{x}_{new})$

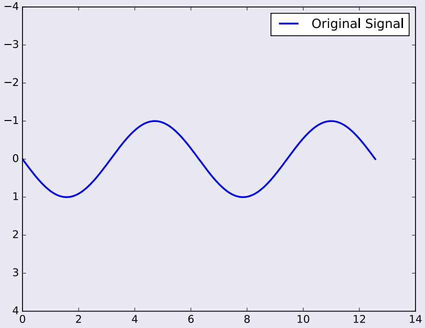
The Machinery in General looks...

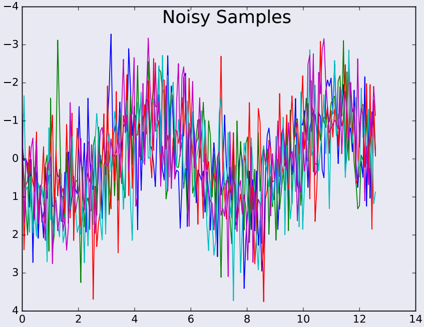


Outline

How do we handle Noise?

Imagine the following signal from $\sin(\theta)$





What if we know the noise?

Given a series of observed samples $\{\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2, ..., \widehat{\boldsymbol{x}}_N\}$ with noise $\epsilon \sim N(0,1)$

We could use our knowledge on the noise, for example additive:

$$\widehat{\boldsymbol{x}}_i = \boldsymbol{x}_i + \epsilon$$

We can use our knowledge of probability to remove such noise

$$E\left[\widehat{\boldsymbol{x}}_i\right] = E\left[\boldsymbol{x}_i + \epsilon\right] = E\left[\boldsymbol{x}_i\right] + E\left[\epsilon\right]$$

Then, because $E\left[\epsilon\right] = 0$

$$E\left[\boldsymbol{x}_i\right] = E\left[\widehat{\boldsymbol{x}}_i\right] \approx \frac{1}{N}\sum_{i=1}^{N} \widehat{\boldsymbol{x}}_i$$

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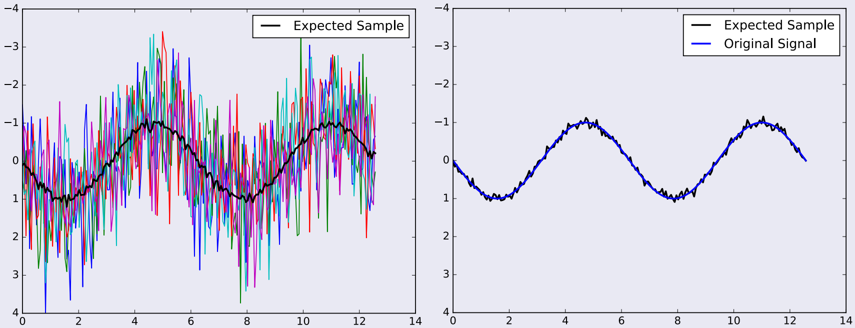
In our example

We have a nice result



Therefore, we have

The Bayesian Models

- They allow to deal with noise from the samples

Quite different from the deterministic models so far

- Unless Samples are Preprocessed to Reduce the Noise

Something that people in area as Control tend to do

- The importance of Filters as Kalman Filters

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Given a Spoken Language

The task is to determine the language that someone is speaking

- Therefore, they try to determine the spoken language based on such learning

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Generative Methods

1. Model class-conditional pdfs and prior probabilities.
2. "Generative" since sampling can generate synthetic data points.

- Mixtures of Gaussians, Mixtures of Experts, Hidden Markov Models (HMM)
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The Rule for classification is the following one

$$P(\omega_i|x) = \frac{P(x|\omega_i) P(\omega_i)}{P(x)} \tag{1}$$

- Remark: Bayes to the next level.

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We have that

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One: If we can observe x.

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We have the following terms...

Likelihood

We call $p\left(\boldsymbol{x}|\omega_i\right)$ the likelihood of ω_i given $\boldsymbol{x}$:

- This indicates that given a category ω_i: If $p\left(\boldsymbol{x}|\omega_i\right)$ is "large", then ω_i is the "likely" class of $\boldsymbol{x}$.

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Prior Probability

It is the known probability of a given class.

Remark: Because, we lack information about this class, we tend to use the uniform distribution.

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The most important term in all this

The factor

$$likelihood \times prior\text{-}information \tag{3}$$

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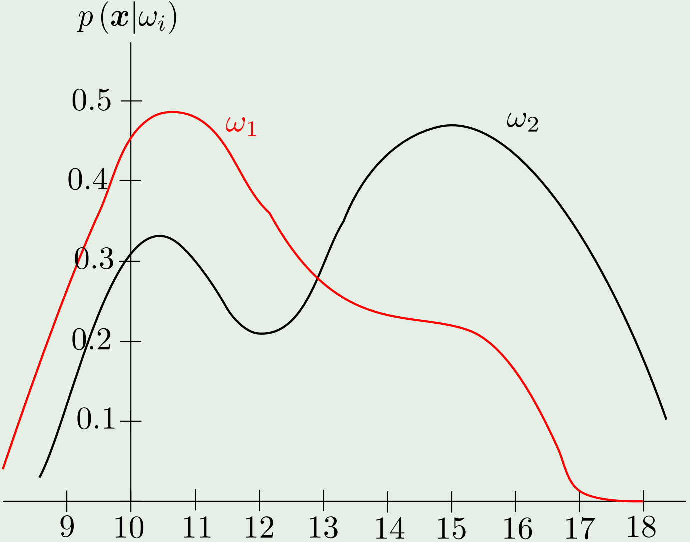
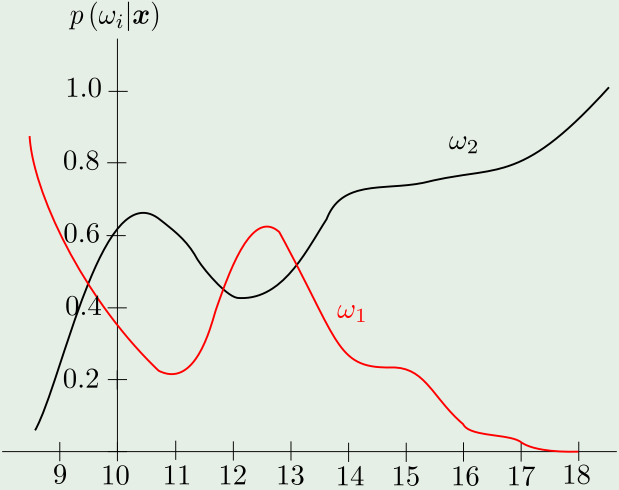
Example

We have the likelihood of two classes



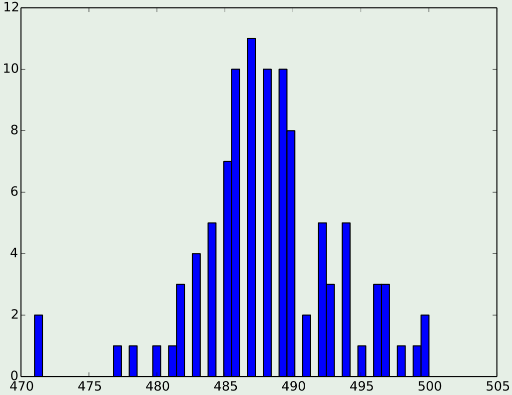
Example

We have the posterior of two classes when $P(\omega_1) = \frac{2}{3}$ and $P(\omega_2) = \frac{1}{3}$



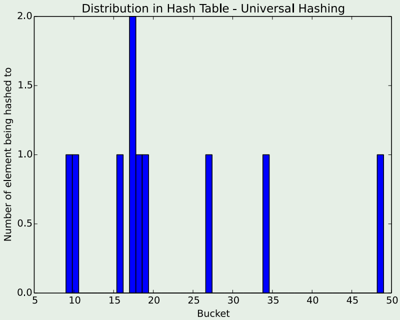
Example of key distribution

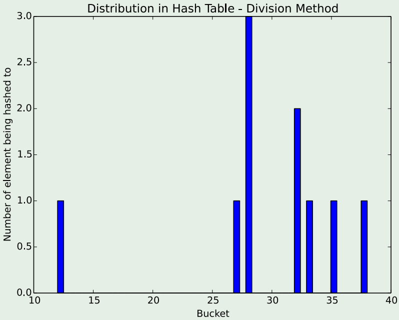
Example, mean = 488.5 and dispersion = 5



Example with 10 keys

Universal Hashing Vs Division Method





Example with 50 keys

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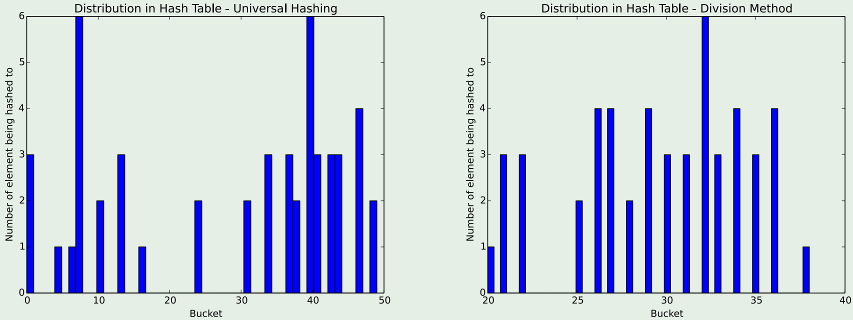

Example with 100 keys

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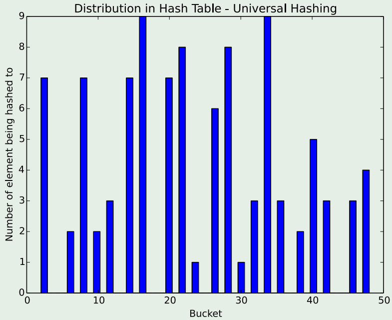

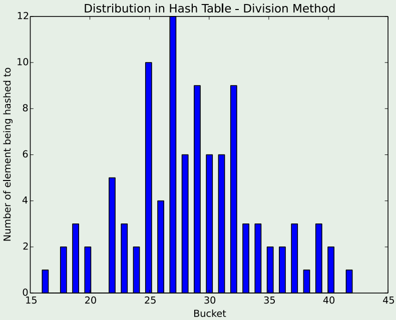

Example with 200 keys

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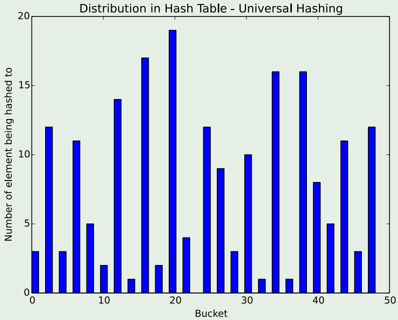

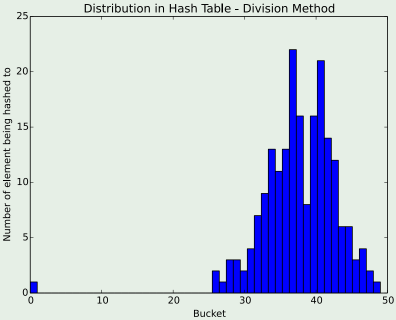

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Naive Bayes Model

In the case of two classes, we can use demarginalization

$$P(\boldsymbol{x}) = \sum_{i=1}^{2} p(\boldsymbol{x}, \omega_i) = \sum_{i=1}^{2} p(\boldsymbol{x}|\omega_i) P(\omega_i) \tag{4}$$

Error in this rule

We have that

$$P\left(error|\boldsymbol{x}\right)=\begin{cases}P\left(\omega_1|\boldsymbol{x}\right) & \text{if we decide } \omega_2\\ P\left(\omega_2|\boldsymbol{x}\right) & \text{if we decide } \omega_1\end{cases} \quad (5)$$

Thus, we have that

$$P\left(error\right)=\int_{-\infty}^{\infty}P\left(error,\boldsymbol{x}\right)d\boldsymbol{x}=\int_{-\infty}^{\infty}P\left(error|\boldsymbol{x}\right)p\left(\boldsymbol{x}\right)d\boldsymbol{x} \quad (6)$$

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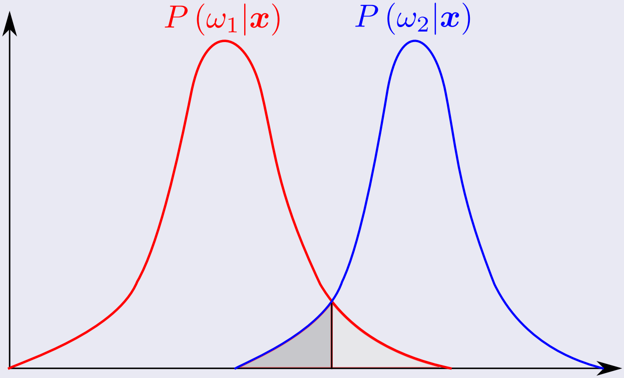
Graphically

We have



$$P(error) = \int_{-\infty}^{\infty} P(error, \boldsymbol{x})\, d\boldsymbol{x}$$

Classification Rule

Thus, we have the Bayes Classification Rule

1. If $P(\omega_1|\boldsymbol{x}) > P(\omega_2|\boldsymbol{x})$ $\boldsymbol{x}$ is classified to ω_1
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What if we remove the normalization factor?

Remember

$$P(\omega_1|\boldsymbol{x}) + P(\omega_2|\boldsymbol{x}) = 1 \tag{7}$$

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We are able to obtain the new Bayes Classification Rule

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We have several cases

If for some $\boldsymbol{x}$ we have $P(\boldsymbol{x}|\omega_1) = P(\boldsymbol{x}|\omega_2)$

The final decision relies completely from the prior probability.

On the Other hand if $P(\omega_1) = P(\omega_2)$, the "state" is equally probable

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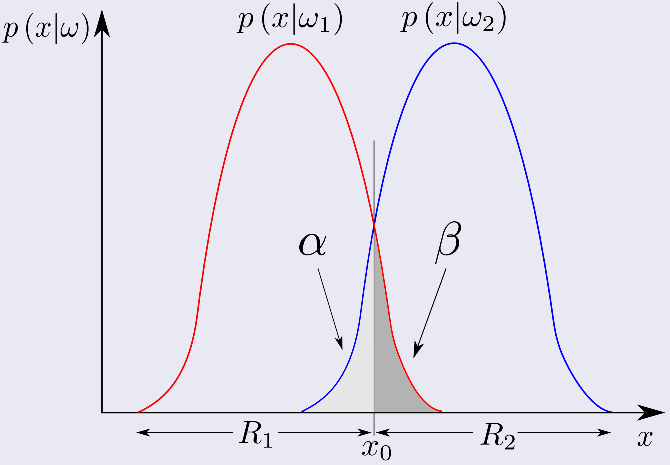
How the Rule looks like

If $P(\omega_1) = P(\omega_2)$ the Rule depends on the term $p(\boldsymbol{x}|\omega_i)$



Error in Naive Bayes

Error in equiprobable classes $p\left(\omega_1\right) = p\left(\omega_2\right) = \frac{1}{2}$

$$
\begin{aligned}
P_e = & \int_{-\infty}^{\infty} P\left(\boldsymbol{x}, error\right) d\boldsymbol{x} \\
= & \int_{-\infty}^{x_0} p\left(x, \omega_2\right) dx + \int_{x_0}^{\infty} p\left(x, \omega_1\right) dx \\
= & \int_{-\infty}^{x_0} p\left(x|\omega_2\right) P\left(\omega_2\right) dx + \int_{x_0}^{\infty} p\left(x|\omega_1\right) P\left(\omega_1\right) dx \\
= & P\left(\omega_2\right) \int_{-\infty}^{x_0} p\left(x|\omega_2\right) dx + P\left(\omega_1\right) \int_{x_0}^{\infty} p\left(x|\omega_1\right) dx \\
= & \frac{1}{2} \int_{-\infty}^{x_0} p\left(x|\omega_2\right) dx + \frac{1}{2} \int_{x_0}^{\infty} p\left(x|\omega_1\right) dx
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&= P\left(\omega_2\right) \int_{-\infty}^{x_0} p\left(x|\omega_2\right) dx + P\left(\omega_1\right) \int_{x_0}^{\infty} p\left(x|\omega_1\right) dx \\
&= \frac{1}{2} \int_{-\infty}^{x_0} p\left(x|\omega_2\right) dx + \frac{1}{2} \int_{x_0}^{\infty} p\left(x|\omega_1\right) dx
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=& \int_{-\infty}^{x_0} p(x|\omega_2)\, P(\omega_2)\, dx + \int_{x_0}^{\infty} p(x|\omega_1)\, P(\omega_1)\, dx \\
=& P(\omega_2) \int_{-\infty}^{x_0} p(x|\omega_2)\, dx + P(\omega_1) \int_{x_0}^{\infty} p(x|\omega_1)\, dx \\
=& \frac{1}{2} \int_{-\infty}^{x_0} p(x|\omega_2)\, dx + \frac{1}{2} \int_{x_0}^{\infty} p(x|\omega_1)\, dx
\end{aligned}
$$

Error in Naive Bayes

Error in equiprobable classes $p(\omega_1) = p(\omega_2) = \frac{1}{2}$

$$
\begin{aligned}
P_e = & \int_{-\infty}^{\infty} P(\boldsymbol{x}, error)\, d\boldsymbol{x} \\
= & \int_{-\infty}^{x_0} p(x, \omega_2)\, dx + \int_{x_0}^{\infty} p(x, \omega_1)\, dx \\
= & \int_{-\infty}^{x_0} p(x|\omega_2)\, P(\omega_2)\, dx + \int_{x_0}^{\infty} p(x|\omega_1)\, P(\omega_1)\, dx \\
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Error in Naive Bayes

Something Notable

Bayesian classifier is optimal with respect to minimizing the classification error probability.

Proof

Step 1

- R_1 be the region of the feature space in which we decide in favor of ω_1
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Thus

$$\begin{aligned} P_e &= P\left(x \in R_2 | \omega_1\right) P\left(\omega_1\right) + P\left(x \in R_1 | \omega_2\right) P\left(\omega_2\right) \\ &= P\left(\omega_1\right) \int_{R_2} p\left(x | \omega_1\right) dx + P\left(\omega_2\right) \int_{R_1} p\left(x | \omega_2\right) dx \end{aligned}$$

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It is more

$$P_e = P(\omega_1) \int\limits_{R_2} \frac{p(\omega_1, x)}{P(\omega_1)} dx + P(\omega_2) \int\limits_{R_1} \frac{p(\omega_2, x)}{P(\omega_2)} dx \tag{9}$$

Finally

$$P_e = \int\limits_{R_2} p(\omega_1|x)\, p(x)\, dx + \int\limits_{R_1} p(\omega_2|x)\, p(x)\, dx \tag{10}$$

Now, we choose the Bayes Classification Rule

$$\begin{aligned} R_1 &: P(\omega_1|x) > P(\omega_2|x) \\ R_2 &: P(\omega_2|x) > P(\omega_1|x) \end{aligned}$$

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$$P(\omega_1) = \int\limits_{R_1} p(\omega_1|x)\,p(x)\,dx + \int\limits_{R_2} p(\omega_1|x)\,p(x)\,dx \tag{11}$$

Now, we have...

$$P(\omega_1) - \int\limits_{R_1} p(\omega_1|x)\,p(x)\,dx = \int\limits_{R_2} p(\omega_1|x)\,p(x)\,dx \tag{12}$$

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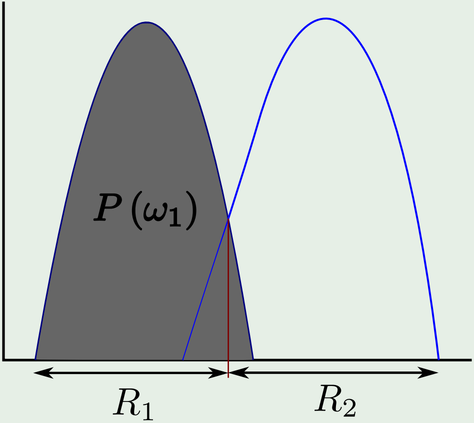
Graphically $P(\omega_1)$: Thanks Edith 2013 Class!!!

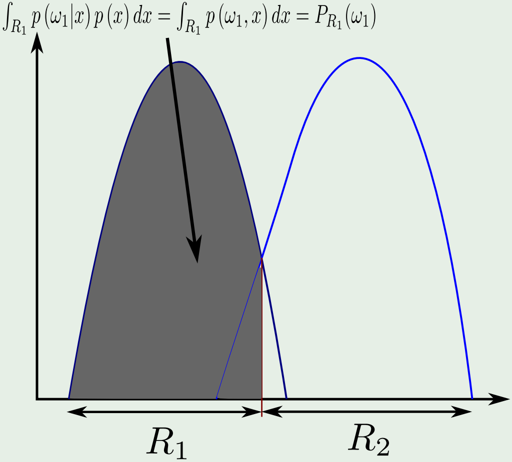
In Gray



Thus we have

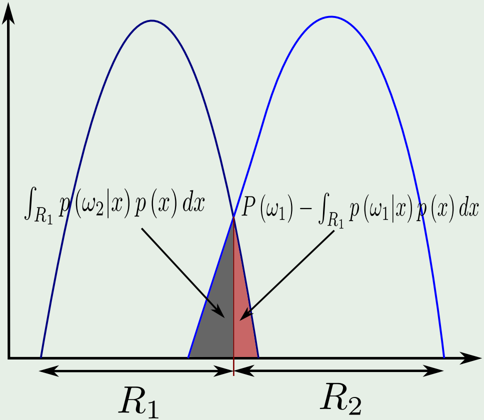
$$\int_{R_1} p(\omega_1|x)\, p(x)\, dx = \int_{R_1} p(\omega_1, x)\, dx = P_{R_1}(\omega_1)$$

Thus



Finally P_e

A great idea Edith!!!



Thus

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$$P_e = P(\omega_1) - \int_{R_1} \left[p(\omega_1|x) - p(\omega_2|x) \right] p(x)\, dx \tag{14}$$

Thus

The probability of error is minimized at the region of space in which $R_1 : P(\omega_1|x) > P(\omega_2|x)$.

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The Naive Bayes Rule minimizes the error.

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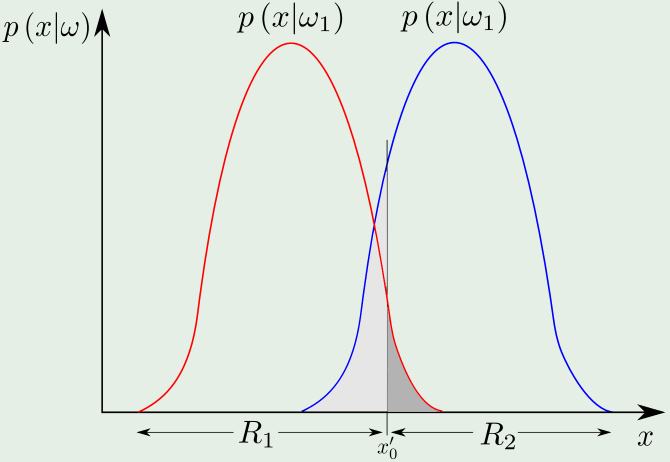
After all!!!

If you choose any other x_0'



Outline

For M classes $\omega_1, \omega_2, ..., \omega_M$

We have that vector $\boldsymbol{x}$ is in ω_i

$$P(\omega_i|\boldsymbol{x}) > P(\omega_j|\boldsymbol{x}) \;\; \forall j \neq i \tag{16}$$

Something Notable

It turns out that such a choice also minimizes the classification error probability.

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Outline

Decision Surface

Because the R_1 and R_2 are contiguous

The separating surface between both of them is described by

$$P\left(\omega_1|x\right) - P\left(\omega_2|x\right) = 0 \tag{17}$$

Thus, we define the decision function as

$$g_{12}\left(x\right) = P\left(\omega_1|x\right) - P\left(\omega_2|x\right) = 0 \tag{18}$$

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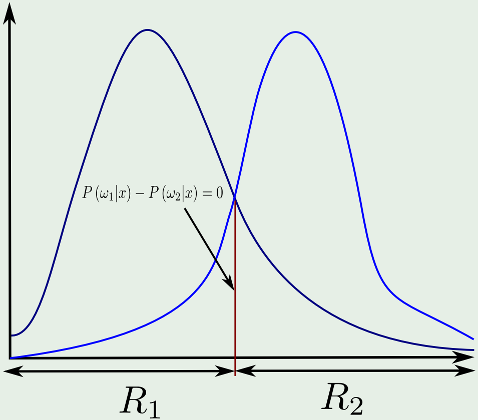
Which decision function for the Naive Bayes

A single number in this case



In general

First

Instead of working with probabilities, we work with an equivalent function of them $g_i(\boldsymbol{x}) = f(P(\omega_i|\boldsymbol{x}))$.

- Classic Example the Monotonically increasing $f(P(\omega_i|\boldsymbol{x})) = \ln P(\omega_i|\boldsymbol{x})$.

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Outline

Gaussian Distribution

We can use the Gaussian distribution

$$p\left(\boldsymbol{x}|\boldsymbol{\omega_i}\right)=\frac{1}{\left(2\pi\right)^{l/2}\left|\Sigma_i\right|^{1/2}}\exp\left\{-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{\mu_i}\right)^T\Sigma_i^{-1}\left(\boldsymbol{x}-\boldsymbol{\mu_i}\right)\right\} \tag{19}$$

Example

Gaussian Distribution

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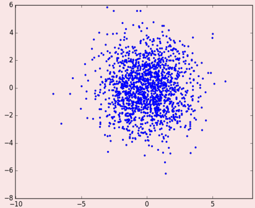
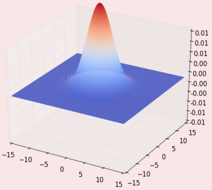
Example

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$





Some Properties

About Σ

It is the covariance matrix between variables.

Thus

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- Symmetric.
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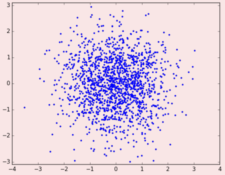

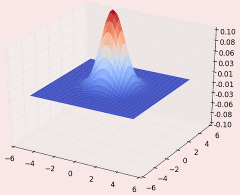

The Covariance Σ as a Rotation

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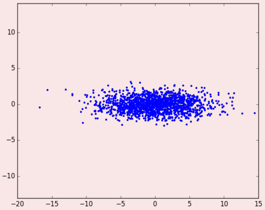

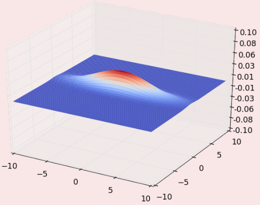

Influence of the Covariance Σ

Look at the following Covariance

$$\Sigma_a = R \Sigma_b R^T \text{ with } R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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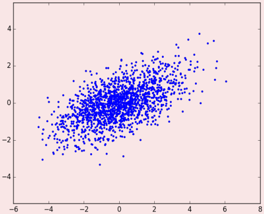

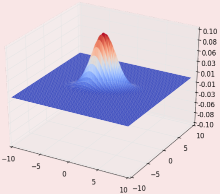

Now For Two Classes

Then, we use the following trick for two Classes $i = 1, 2$

We know that the pdf of correct classification is
$p\left(x, \omega_1\right) = p\left(x|\omega_i\right) P\left(\omega_i\right)$!!!

Thus

It is possible to generate the following decision function:

$$g_i\left(x\right) = \ln\left[p\left(x|\omega_i\right) P\left(\omega_i\right)\right] = \ln p\left(x|\omega_i\right) + \ln P\left(\omega_i\right) \tag{20}$$

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We can work one of the possible decision surfaces

Assume first that $\Sigma_i = \sigma^2 I$

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For Example

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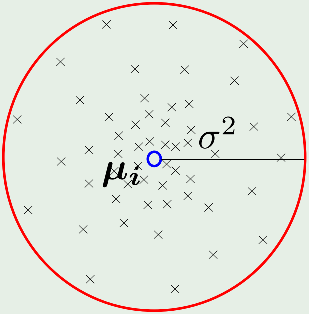

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$$|\Sigma_i| = \sigma^{2d} \text{ and } \Sigma_i^{-1} = \left(\frac{1}{\sigma^2}\right) I$$

Something Notable

- Gaussian Multivariate function after the log

$$g_i(x) = -\frac{1}{2}(x - \mu_i)^T \Sigma_i^{-1} (x - \mu_i) + \ln P(\omega_i) - \frac{d}{2}\ln 2\pi - \frac{1}{2}\ln|\Sigma_i|$$

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We have the following discriminant functions

$$g_i(\boldsymbol{x}) = -\frac{\overbrace{(\boldsymbol{x}-\boldsymbol{\mu}_i)^T(\boldsymbol{x}-\boldsymbol{\mu}_i)}^{\|\boldsymbol{x}-\boldsymbol{\mu}_i\|^2}}{2\sigma^2} + \ln P(\omega_i) \tag{22}$$

Then, we have that

$$g_i(\boldsymbol{x}) = -\frac{1}{2\sigma^2}\left[\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{\mu}_i^T\boldsymbol{x} + \boldsymbol{\mu}_i^T\boldsymbol{\mu}_i\right] + \ln P(\omega_i)$$

Then

We have the following discriminant functions

$$g_i(\boldsymbol{x}) = -\frac{\overbrace{(\boldsymbol{x}-\boldsymbol{\mu_i})^T(\boldsymbol{x}-\boldsymbol{\mu_i})}^{\|\boldsymbol{x}-\boldsymbol{\mu_i}\|^2}}{2\sigma^2} + \ln P(\omega_i) \tag{22}$$

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We can then...

Do you notice that $\boldsymbol{x}^T\boldsymbol{x}$ is actually the same for all g_i?

Then, we can ignore that term thus, we get

$$g_i(\boldsymbol{x}) = \underbrace{\frac{1}{\sigma^2}\boldsymbol{\mu_i}}_{\boldsymbol{w}_i^T}{}^T\boldsymbol{x} - \frac{1}{2\sigma^2}\boldsymbol{\mu_i}^T\underbrace{\boldsymbol{\mu_i}}_{w_{i0}} + \ln P(\omega_i)$$

Or if you want

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$$g_i(\boldsymbol{x}) = \boldsymbol{w}_i^T\boldsymbol{x} + w_{i0}$$

Outline

Given a series of classes $\omega_1, \omega_2, ..., \omega_M$

We assume for each class ω_j

The samples are drawn independently according to the probability law $p(\boldsymbol{x}|\omega_j)$

We call those samples as

i.i.d. — independent identically distributed random variables.

We assume in addition

$p(\boldsymbol{x}|\omega_j)$ has a known parametric form with vector θ_j of parameters.

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Given a series of classes $\omega_1, \omega_2, ..., \omega_M$

For example

$$p(\boldsymbol{x}|\omega_j) \sim N\left(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j\right) \tag{23}$$

In our case

We will assume that there is no dependence between classes!!!

Given a series of classes $\omega_1, \omega_2, ..., \omega_M$

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Now

Suppose that ω_j contains n samples $\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n$

$$p\left(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n|\boldsymbol{\theta}_j\right) = \prod_{j=1}^{n} p\left(\boldsymbol{x}_j|\boldsymbol{\theta}_j\right) \tag{24}$$

We can see then the function $p\left(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n|\boldsymbol{\theta}_j\right)$ as a function of

$$L\left(\boldsymbol{\theta}_j\right) = \prod_{j=1}^{n} p\left(\boldsymbol{x}_j|\boldsymbol{\theta}_j\right) \tag{25}$$

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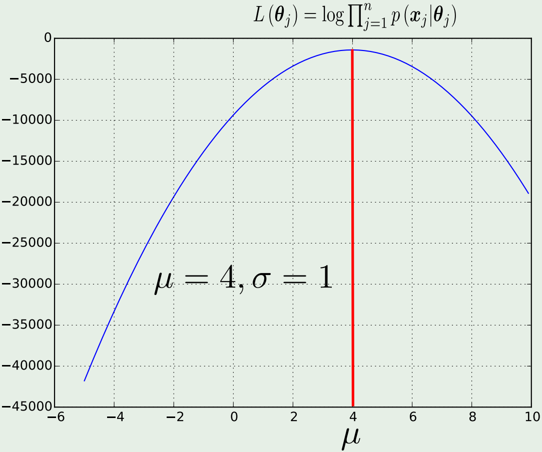
Example

$L\left(\boldsymbol{\theta}_j\right) = \log \prod_{j=1}^{n} p\left(\boldsymbol{x}_j | \boldsymbol{\theta}_j\right)$



Outline

Maximum Likelihood on a Gaussian

Then, using the log!!!

$$\ln L\left(\omega_i\right) = -\frac{n}{2}\ln\left|\Sigma_i\right| - \frac{1}{2}\left[\sum_{j=1}^{n}\left(\boldsymbol{x_j}-\boldsymbol{\mu_i}\right)^T\Sigma_i^{-1}\left(\boldsymbol{x_j}-\boldsymbol{\mu_i}\right)\right] + c_2 \quad (26)$$

We know that

$$\frac{dx^T Ax}{dx} = Ax + A^T x,\ \ \frac{dAx}{dx} = A \quad (27)$$

Thus, we expand equation26

$$-\frac{n}{2}\ln\left|\Sigma_i\right| - \frac{1}{2}\sum_{j=1}^{n}\left[x_j{}^T\Sigma_i^{-1}x_j - 2x_j{}^T\Sigma_i^{-1}\mu_i + \mu_i{}^T\Sigma_i^{-1}\mu_i\right] + c_2 \quad (28)$$

Maximum Likelihood on a Gaussian

Then, using the log!!!

$$\ln L\left(\omega_i\right) = -\frac{n}{2}\ln\left|\Sigma_i\right| - \frac{1}{2}\left[\sum_{j=1}^{n}\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)^T \Sigma_i^{-1}\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)\right] + c_2 \tag{26}$$

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Maximum Likelihood

Then

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\begin{aligned}
\frac{\partial \ln L(\omega_i)}{\partial \boldsymbol{\mu}_i} &= \sum_{j=1}^{n} \Sigma_i^{-1} \left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right) = 0 \\
n\Sigma_i^{-1}\left[-\boldsymbol{\mu}_i + \frac{1}{n}\sum_{j=1}^{n} \boldsymbol{x}_j\right] &= 0 \\
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Maximum Likelihood

Then, we derive with respect to Σ_i

For this we use the following tricks:

1. $\frac{\partial \log|\Sigma|}{\partial \Sigma^{-1}} = -\frac{1}{|\Sigma|} \cdot |\Sigma| \left(\Sigma\right)^T = -\Sigma$
2. $\frac{\partial Tr[AB]}{\partial A} = \frac{\partial Tr[BA]}{\partial A} = B^T$
3. Trace(of a number)=the number
4. $Tr(A^T B) = Tr\left(BA^T\right)$

Thus

$$f\left(\Sigma_i\right) = -\frac{n}{2}\ln\left|\Sigma_I\right| - \frac{1}{2}\sum_{j=1}^{n}\left[\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)^T \Sigma_i^{-1}\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)\right] + c_1 \tag{29}$$

Maximum Likelihood

Thus

$$f\left(\Sigma_i\right) = -\frac{n}{2}\ln\left|\Sigma_i\right| - \frac{1}{2}\sum_{j=1}^{n}\left[Trace\left\{\left(\boldsymbol{x_j}-\boldsymbol{\mu_i}\right)^T\Sigma_i^{-1}\left(\boldsymbol{x_j}-\boldsymbol{\mu_i}\right)\right\}\right] + c_1 \quad (30)$$

Tricks!!!

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Maximum Likelihood

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Derivative with respect to Σ

$$\frac{\partial f\left(\Sigma_i\right)}{\partial \Sigma_i} = \frac{n}{2}\Sigma_i - \frac{1}{2}\sum_{j=1}^{n}\left[\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)^T\right]^T \quad (32)$$

Thus, when making it equal to zero

$$\hat{\Sigma}_i = \frac{1}{n}\sum_{j=1}^{n}\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)\left(\boldsymbol{x_j} - \boldsymbol{\mu_i}\right)^T \quad (33)$$

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Therefore

Step 1 - Assume a Gaussian Distribution over each class

- The So Called Model Selection

Step 2

- Adjust the Gaussian Distribution, for each class, using the previous Maximum Likelihood

Step 3

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\begin{aligned}
R_1 &: P(\omega_1|x) > P(\omega_2|x) \\
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Outline

In the case of Bayesian Model

We have

$$P\left(Y_n = i|\boldsymbol{x}_n\right) = \frac{P\left(\boldsymbol{x}_n|Y_n = i\right) P\left(Y_n = i\right)}{P\left(\boldsymbol{x}_n\right)}$$

In the Generative Model

- We model two distribution $P\left(x_n|Y_n = 1\right)$ and $P\left(Y_n = i\right)$

In the Discriminative Model

- We model a single distribution $P\left(Y_n = i\right)$

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We have

- In the Generative Model, we discover the distribution from X and Y

Therefore

Although discriminative models tend to be faster and less complex, they cannot model the joint $P(X,Y)$.

Thus

- We have a decision problem
  - Do we want to know the joint distribution?

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Outline

Introduction

We go back to the Bayesian Rule

$$p\left(\Theta|\mathcal{X}\right)=\frac{p\left(\mathcal{X}|\Theta\right)p\left(\Theta\right)}{p\left(\mathcal{X}\right)} \tag{34}$$

We now seek that value for Θ, called $\widehat{\Theta}_{MAP}$

It allows to maximize the posterior $p\left(\Theta|\mathcal{X}\right)$

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Development of the solution

We look to maximize $\widehat{\Theta}_{MAP}$

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$P\left(\mathcal{X}\right)$ can be removed because it has no functional relation with Θ.

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We look to maximize $\widehat{\Theta}_{MAP}$

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We can make this easier

Use logarithms

$$\widehat{\Theta}_{MAP} = \underset{\Theta}{\mathrm{argmax}} \left[\sum_{x_i \in \mathcal{X}} \log p\left(x_i|\Theta\right) + \log p\left(\Theta\right) \right] \tag{35}$$

Outline

What can we do?

We can specify a distribution

Then, learn the parameters

Remember the Bayesian Rule

$$p(\Theta|\mathcal{X}) = \frac{p(\mathcal{X}|\Theta)\, p(\Theta)}{p(\mathcal{X})} \quad (36)$$

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Therefore

We can use this idea of maximizing the posterior

To obtain the distribution through the Maximum a Posteriori

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What Does the MAP Estimate Get?

Something Notable

The MAP estimate allows us to inject into the estimation calculation our prior beliefs regarding the parameters values in Θ.

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The MAP estimate allows us to inject into the estimation calculation our prior beliefs regarding the parameters values in Θ.

For example

Let's conduct N independent trials of the following Bernoulli experiment with q parameter:

- We will ask each individual we run into in the hallway whether they will vote PRI or PAN in the next presidential election.

What Does the MAP Estimate Get?

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First the Maximum Likelihood Estimate

Samples

$$\mathcal{X} = \left\{ x_i = \begin{cases} PAN \\ PRI \end{cases} \quad i = 1, ..., N \right\} \tag{38}$$

The log likelihood function

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The log likelihood function

$$\begin{aligned}
\log\ p\left(\mathcal{X}|q\right) &= \sum_{i=1}^{N} \log\ p\left(x_i|q\right) \\
&= \sum_{i} \log\ p\left(x_i = PRI|q\right) + ... \\
& \sum_{i} \log\ p\left(x_i = PAN|1-q\right) \\
&= n_{PRI} \log\left(q\right) + \left(N - n_{PRI}\right) \log\left(1-q\right)
\end{aligned}$$

Where n_{PRI} are the numbers of individuals who are planning to vote PRI this fall

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We use our classic tricks

By setting

$$\mathcal{L} = \log \; p(\mathcal{X}|q) \tag{39}$$

We have that

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \tag{40}$$

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$$\frac{n_{PRI}}{q} - \frac{(N - n_{PRI})}{(1-q)} = 0 \tag{41}$$

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Final Solution of ML

We get

$$\widehat{q}_{PRI} = \frac{n_{PRI}}{N} \tag{42}$$

Thus

If we say that $N = 20$ and if 12 are going to vote PRI, we get $\widehat{q}_{PRI} = 0.6$.

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Building the MAP estimate

Obviously we need a prior belief distribution

We have the following constraints:

- The prior for q must be zero outside the $[0, 1]$ interval.
- Within the $[0, 1]$ interval, we are free to specify our beliefs in any way we wish.
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What prior distribution can we use?

We could use a Beta distribution being parametrized by two values α and β

$$p(q) = \frac{1}{B(\alpha, \beta)} q^{\alpha - 1} (1 - q)^{\beta - 1}. \tag{43}$$

Where

We have $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function where Γ is the generalization of the notion of factorial in the case of the real numbers.

Properties

When both the $\alpha, \beta > 0$ then the beta distribution has its mode (Maximum value) at

$$\frac{\alpha - 1}{\alpha + \beta - 2}. \tag{44}$$

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We can choose $\alpha = \beta$ so the beta prior peaks at 0.5.

As a further expression of our belief

We make the following choice $\alpha = \beta = 5$.

Why? Look at the variance of the beta distribution

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We have a variance with $\alpha = \beta = 5$

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Now, our MAP estimate for $\widehat{p}_{MAP}$...

We have then

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Plugging back the ML

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The log of $p(q)$

We have that

$$\log p(q) = (\alpha - 1)\log q + (\beta - 1)\log(1 - q) - \log B(\alpha, \beta) \tag{49}$$

Now taking the derivative with respect to p, we get

$$\frac{n_{PRI}}{q} - \frac{(N - n_{PRI})}{(1 - q)} - \frac{\beta - 1}{1 - q} + \frac{\alpha - 1}{q} = 0 \tag{50}$$

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Now

With $N = 20$ with $n_{PRI} = 12$ and $\alpha = \beta = 5$

$$\widehat{q}_{MAP} = 0.571$$

Outline

Properties

First

MAP estimation "pulls" the estimate toward the prior.

Second

The more focused our prior belief, the larger the pull toward the prior.

Example

If $\alpha = \beta =$equal to large value

- It will make the MAP estimate to move closer to the prior.

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In the expression we derived for $\widehat{q}_{MAP}$, the parameters α and β play a "smoothing" role vis-a-vis the measurement n_{PRI}.

Fourth

Since we referred to q as the parameter to be estimated, we can refer to α and β as the hyper-parameters in the estimation calculations.

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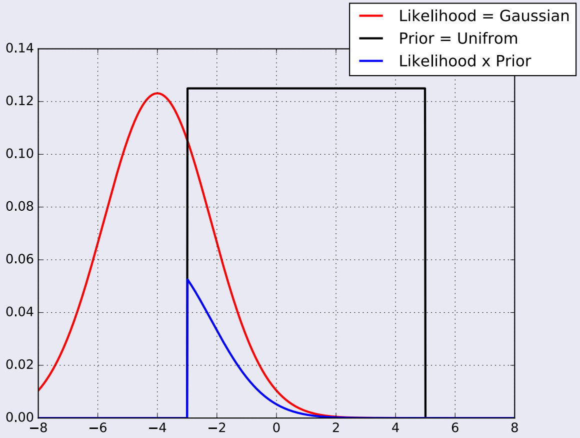
Basically the MAP

It is using the power of Likelihood $\times$ Prior to obtain more information from the data



Beyond simple derivation

In the previous technique

We took an logarithm of the **likelihood** $\times$ **the prior** to obtain a function that can be derived in order to obtain each of the parameters to be estimated.

What if we cannot derive the likelihood $\times$ the prior?

For example when we have something like $|\theta_i|$.

We can try the following

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Exercises

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Chapter 3

- 3.1, 3.2, 3.3, 3.13

Theodoridis

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- 2.5, 2.7, 2.10, 2.12, 2.14, 2.17

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