Mathematics for Artificial Intelligence

Square Matrices and Related Matters

Andres Mendez-Vazquez

March 2, 2020

Outline

Cinvestav

Outline

Cinvestav

Square Matrices

Observation

Square matrices are the only matrices that can have inverses.

Further

In a system of linear algebraic equations:

1. If the number of equations equals the number of unknowns
2. Then the associated coefficient matrix A is square.

Now, use the Gauss-Jordan

What can happen?

Square Matrices

Observation

Square matrices are the only matrices that can have inverses.

Further

In a system of linear algebraic equations:

1. If the number of equations equals the number of unknowns
2. Then the associated coefficient matrix A is square.

Now, use the Gauss-Jordan

What can happen?

Square Matrices

Observation

Square matrices are the only matrices that can have inverses.

Further

In a system of linear algebraic equations:

1. If the number of equations equals the number of unknowns
2. Then the associated coefficient matrix A is square.

Now, use the Gauss-Jordan

What can happen?

Outline

Cinvestav

We have two possibilities

First case

The Gauss-Jordan form for $A_{n \times n}$ is the $n \times n$ identity matrix I_n

Second case

The Gauss-Jordan form for A has at least one row of zeros

We have two possibilities

First case

The Gauss-Jordan form for $A_{n \times n}$ is the $n \times n$ identity matrix I_n

Second case

The Gauss-Jordan form for A has at least one row of zeros

Explanation

In the first case

We can show that A is invertible.

How? Do you remember?

$$E_k E_{k-1} \ldots E_2 E_1 A = I,$$

Setting $B = E_k E_{k-1} \ldots E_2 E_1$

We have $BA = I$ therefore $B = A^{-1}$

Explanation

In the first case

We can show that A is invertible.

How? Do you remember?

$$E_k E_{k-1} ... E_2 E_1 A = I,$$

Setting $B = E_k E_{k-1} ... E_2 E_1$

We have $BA = I$ therefore $B = A^{-1}$

Explanation

In the first case

We can show that A is invertible.

How? Do you remember?

$$E_k E_{k-1} ... E_2 E_1 A = I,$$

Setting $B = E_k E_{k-1} ... E_2 E_1$

We have $BA = I$ therefore $B = A^{-1}$

Furthermore

We can build the following matrix

$$E_1^{-1}E_2^{-1}\cdots E_k^{-1}$$

Then

$$\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)BA=\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)I$$

Thus

$$\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)\left(E_kE_{k-1}\ldots E_2E_1\right)A=\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)$$

Furthermore

We can build the following matrix

$$E_1^{-1}E_2^{-1}\cdots E_k^{-1}$$

Then

$$\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)BA=\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)I$$

Thus

$$\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)\left(E_kE_{k-1}\ldots E_2E_1\right)A=\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)$$

Furthermore

We can build the following matrix

$$E_1^{-1}E_2^{-1}\cdots E_k^{-1}$$

Then

$$\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)BA=\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)I$$

Thus

$$\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)\left(E_kE_{k-1}...E_2E_1\right)A=\left(E_1^{-1}E_2^{-1}\cdots E_k^{-1}\right)$$

Therefore

We have

$$A = \left(E_1^{-1} E_2^{-1} \cdots E_k^{-1}\right)$$

Theorem

The following are equivalent:

1. The square matrix A is invertible.
2. The Gauss-Jordan or reduced echelon form of A is the identity matrix.
3. A can be written as a product of elementary matrices.

Therefore

We have

$$A = \left(E_1^{-1} E_2^{-1} \cdots E_k^{-1}\right)$$

Theorem

The following are equivalent:

1. The square matrix A is invertible.
2. The Gauss-Jordan or reduced echelon form of A is the identity matrix.
3. Acan be written as a product of elementary matrices.

The Second Case

The Gauss-Jordan form of $A_{n\times n}$

- It can only have at most n leading entries.

If the Gauss-Jordan form of A is not I

- We have something quite different

Then the GJ form has $n-1$ or fewer leading entries

- Therefore, it has at least one row of zeros.

Cinvestav

The Second Case

The Gauss-Jordan form of $A_{n\times n}$

- It can only have at most n leading entries.

If the Gauss-Jordan form of A is not I

- We have something quite different

Then the GJ form has $n-1$ or fewer leading entries

- Therefore, it has at least one row of zeros.

Cinvestav

The Second Case

The Gauss-Jordan form of $A_{n\times n}$

- It can only have at most n leading entries.

If the Gauss-Jordan form of A is not I

- We have something quite different

Then the GJ form has $n-1$ or fewer leading entries

- Therefore, it has at least one row of zeros.

Example

Given

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

What do we need to do?

Look at the blackboard...

Therefore

$$A^{-1} = E_4 E_3 E_2 E_1$$

Example

Given

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

What do we need to do?

Look at the blackboard...

Therefore

$$A^{-1} = E_4E_3E_2E_1$$

Example

Given

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

What do we need to do?

Look at the blackboard...

Therefore

$$A^{-1} = E_4 E_3 E_2 E_1$$

Cinvestav

Outline

Cinvestav

If A is invertible a.k.a. full rank

Equation $A\boldsymbol{x} = \boldsymbol{y}$ has the unique solution

$$A\boldsymbol{x} = \boldsymbol{y} \Leftrightarrow A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{y} \Leftrightarrow \boldsymbol{x} = A^{-1}\boldsymbol{y}$$

- There are non-trivial solutions to $Ax = 0$

- Solutions to the system exist, but there are infinitely many

If A is invertible a.k.a. full rank

Equation $A\boldsymbol{x} = \boldsymbol{y}$ has the unique solution

$$A\boldsymbol{x} = \boldsymbol{y} \Leftrightarrow A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{y} \Leftrightarrow \boldsymbol{x} = A^{-1}\boldsymbol{y}$$

If A is not invertible

- Then there is at least one free variable.
- There are non-trivial solutions to $A\boldsymbol{x} = \boldsymbol{0}$.

- Solutions to the system exist, but there are infinitely many

If A is invertible a.k.a. full rank

Equation $A\boldsymbol{x} = \boldsymbol{y}$ has the unique solution

$$A\boldsymbol{x} = \boldsymbol{y} \Leftrightarrow A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{y} \Leftrightarrow \boldsymbol{x} = A^{-1}\boldsymbol{y}$$

If A is not invertible

- Then there is at least one free variable.
- There are non-trivial solutions to $A\boldsymbol{x} = \mathbf{0}$.

If $\boldsymbol{y} \neq \mathbf{0}$

- Either $A\boldsymbol{x} = \boldsymbol{y}$ is inconsistent.
- Solutions to the system exist, but there are infinitely many.

Cinvestav

If A is invertible a.k.a. full rank

Equation $A\boldsymbol{x} = \boldsymbol{y}$ has the unique solution

$$A\boldsymbol{x} = \boldsymbol{y} \Leftrightarrow A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{y} \Leftrightarrow \boldsymbol{x} = A^{-1}\boldsymbol{y}$$

If A is not invertible

- Then there is at least one free variable.
- There are non-trivial solutions to $A\boldsymbol{x} = \mathbf{0}$.

If $\boldsymbol{y} \neq \mathbf{0}$

- Either $A\boldsymbol{x} = \boldsymbol{y}$ is inconsistent.
- Solutions to the system exist, but there are infinitely many.

If A is invertible a.k.a. full rank

Equation $A\boldsymbol{x} = \boldsymbol{y}$ has the unique solution

$$A\boldsymbol{x} = \boldsymbol{y} \Leftrightarrow A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{y} \Leftrightarrow \boldsymbol{x} = A^{-1}\boldsymbol{y}$$

If A is not invertible

- Then there is at least one free variable.
- There are non-trivial solutions to $A\boldsymbol{x} = \mathbf{0}$.

If $\boldsymbol{y} \neq \mathbf{0}$

- Either $A\boldsymbol{x} = \boldsymbol{y}$ is inconsistent.
- Solutions to the system exist, but there are infinitely many.

Outline

Cinvestav

Something Quite Important

We have done something important

It leads immediately to an algorithm for constructing the inverse of A.

Observation

Suppose $B_{n\times p}$ is another matrix with the same number of rows as $A_{n\times n}$

Then

$$C = (A|B)_{n\times(n+p)}$$

Something Quite Important

We have done something important

It leads immediately to an algorithm for constructing the inverse of A.

Observation

Suppose $B_{n\times p}$ is another matrix with the same number of rows as $A_{n\times n}$

Then

$$C = (A|B)_{n\times(n+p)}$$

Cinvestav

Something Quite Important

We have done something important

It leads immediately to an algorithm for constructing the inverse of A.

Observation

Suppose $B_{n\times p}$ is another matrix with the same number of rows as $A_{n\times n}$

Then

$$C = (A|B)_{n\times(n+p)}$$

Then

It is obvious

$$EC = (EA|EB)_{n \times (n+p)}$$

Where

- EA is a $n \times n$ matrix.
- EB is a $n \times p$ matrix

Then

It is obvious

$$EC = (EA|EB)_{n\times(n+p)}$$

Where

- EA is a $n \times n$ matrix.
- EB is a $n \times p$ matrix

Cinvestav

Algorithm

Form the partitioned matrix

$$C = (A|I)$$

Apply the Gauss-Jordan reduction

$$E_k E_{k-1} \cdots E_1 (A|I) = (E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I)$$

Therefore, if A is invertible

$$(E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I) = \left(I | A^{-1}\right)$$

Algorithm

Form the partitioned matrix

$$C = (A|I)$$

Apply the Gauss-Jordan reduction

$$E_k E_{k-1} \cdots E_1 (A|I) = (E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I)$$

Therefore, if A is invertible

$$(E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I) = \left(I | A^{-1}\right)$$

Cinvestav

Algorithm

Form the partitioned matrix

$$C = (A|I)$$

Apply the Gauss-Jordan reduction

$$E_k E_{k-1} \cdots E_1 (A|I) = (E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I)$$

Therefore, if A is invertible

$$(E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1 I) = \left(I | A^{-1}\right)$$

Cinvestav

Remark

The individual factors in the product of A^{-1} are not unique

They depend on how we do the row reduction.

Outline

Cinvestav

Before the Matrix we had the Determinant

You have seen determinants in your classes long ago

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow det\left(A\right) = ad - bc$$

What about

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then

$$\begin{aligned} det\left(A\right) = & a_{11}a_{22}a_{33} + a_{12}a_{21}a_{32} + a_{12}a_{23}a_{31} - \dots \\ & a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

Before the Matrix we had the Determinant

You have seen determinants in your classes long ago

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow det\left(A\right)=ad-bc$$

What about

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then

$$det\left(A\right)=a_{11}a_{22}a_{33}+a_{12}a_{21}a_{32}+a_{12}a_{23}a_{31}-\dots$$
$$a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}-a_{13}a_{22}a_{31}$$

Before the Matrix we had the Determinant

You have seen determinants in your classes long ago

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow det\left(A\right) = ad - bc$$

What about

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then

$$\begin{aligned} det\left(A\right) = & a_{11}a_{22}a_{33} + a_{12}a_{21}a_{32} + a_{12}a_{23}a_{31} - ... \\ & a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} \end{aligned}$$

Recursive Definition

Definition

Let A be a $n \times n$ matrix. Then the determinant of A is defined as follow:

$$det\left(A\right) = \begin{cases} a_{11} & \text{if } n = 1 \\ \sum_{i=1}^{n} a_{i1} A_{i1} & \text{if } n > 1 \end{cases}$$

Where

A_{ij} is the (i, j) −cofactor where

$$A_{ij} = (-1)^{i+j}\, det\left(M_{ij}\right)$$

Here

M_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing its i^{th} row and j^{th} column.

Recursive Definition

Definition

Let A be a $n \times n$ matrix. Then the determinant of A is defined as follow:

$$det\,(A) = \begin{cases} a_{11} & \text{if } n = 1 \\ \sum_{i=1}^{n} a_{i1} A_{i1} & \text{if } n > 1 \end{cases}$$

Where

A_{ij} is the (i,j) −cofactor where

$$A_{ij} = (-1)^{i+j}\, det\,(M_{ij})$$

Here

M_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing its i^{th} row and j^{th} column.

Recursive Definition

Definition

Let A be a $n \times n$ matrix. Then the determinant of A is defined as follow:

$$det\left(A\right) = \begin{cases} a_{11} & \text{if } n = 1 \\ \sum_{i=1}^{n} a_{i1} A_{i1} & \text{if } n > 1 \end{cases}$$

Where

A_{ij} is the (i, j) $-$cofactor where

$$A_{ij} = (-1)^{i+j} \, det\left(M_{ij}\right)$$

Here

M_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing its i^{th} row and j^{th} column.

Example

We have

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Outline

Cinvestav

Problems Will Robinson!!!

We have that for a $A_{n \times n}$

$det\left(A\right)$ has n factorials $(n!)$terms

Problems

The fastest computer of the world will take forever to finish

Problems Will Robinson!!!

We have that for a $A_{n \times n}$

$det\left(A\right)$ has n factorials $(n!)$terms

Problems

The fastest computer of the world will take forever to finish

Cinvestav

Thus, we have some problems with that

Floating point arithmetic

It is not at all the same thing as working with real numbers.

Representation

$$x = (d_1 d_2 d_3 \cdots d_n) \times 2^{a_1 a_2 \cdots a_m} \quad (1)$$

The problem is at the round off

When we do a calculation on a computer, we almost never get the right answer.

Thus, we have some problems with that

Floating point arithmetic

It is not at all the same thing as working with real numbers.

Representation

$$x = (d_1 d_2 d_3 \cdots d_n) \times 2^{a_1 a_2 \cdots a_m} \tag{1}$$

The problem is at the round off

When we do a calculation on a computer, we almost never get the right answer.

Cinvestav

Thus, we have some problems with that

Floating point arithmetic

It is not at all the same thing as working with real numbers.

Representation

$$x = (d_1 d_2 d_3 \cdots d_n) \times 2^{a_1 a_2 \cdots a_m} \quad (1)$$

The problem is at the round off

When we do a calculation on a computer, we almost never get the right answer.

Another Problems

We would love the floating points to be represented uniformly

The floating point numbers are distributed logarithmically which is quite different from the even distribution of the rationals.

- 12 multiplications and 5 additions to compute the [illegible] determinant
- 72 multiplications and 23 additions to

Another Problems

We would love the floating points to be represented uniformly

The floating point numbers are distributed logarithmically which is quite different from the even distribution of the rationals.

Computations do not scale well

- 2 multiplications and 1 addition to compute the 2×2 determinant
- 12 multiplications and 5 additions to compute the 3×3 determinant
- 72 multiplications and 23 additions to

Cinvestav

Another Problems

We would love the floating points to be represented uniformly

The floating point numbers are distributed logarithmically which is quite different from the even distribution of the rationals.

Computations do not scale well

- 2 multiplications and 1 addition to compute the 2×2 determinant
- 12 multiplications and 5 additions to compute the 3×3 determinant
- 72 multiplications and 23 additions to

Another Problems

We would love the floating points to be represented uniformly

The floating point numbers are distributed logarithmically which is quite different from the even distribution of the rationals.

Computations do not scale well

- 2 multiplications and 1 addition to compute the 2×2 determinant
- 12 multiplications and 5 additions to compute the 3×3 determinant
- 72 multiplications and 23 additions to

Cinvestav

Outline

Cinvestav

Therefore

How do we avoid to get us into problems?

We need to define our determinant as a different structure....

Definition

The determinant of A is a real-valued function of the rows of A which we write as

$$det\left(A\right) = det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_n\right)$$

Therefore

How do we avoid to get us into problems?

We need to define our determinant as a different structure....

Definition

The determinant of A is a real-valued function of the rows of A which we write as

$$det\left(A\right) = det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_{\boldsymbol{n}}\right)$$

Properties

Multiplying a row by the constant c multiplies the determinant by c

$$det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., c\boldsymbol{r}_i, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) = c\,det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_i, ..., \boldsymbol{r}_{\boldsymbol{n}}\right)$$

If row i is the sum of the two row vectors x and y

$$det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{x}+\boldsymbol{y}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) = det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{x}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) + ...$$
$$det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{y}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right)$$

Meaning

The determinant is a linear function of each row.

Properties

Multiplying a row by the constant c multiplies the determinant by c

$$det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., c\boldsymbol{r}_i, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) = c\,det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_i, ..., \boldsymbol{r}_{\boldsymbol{n}}\right)$$

If row i is the sum of the two row vectors x and y

$$\begin{aligned} det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{x}+\boldsymbol{y}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) =& det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{x}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) + ... \\ & det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{y}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) \end{aligned}$$

Meaning

The determinant is a linear function of each row.

Properties

Multiplying a row by the constant c multiplies the determinant by c

$$det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., c\boldsymbol{r}_i, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) = c\,det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_i, ..., \boldsymbol{r}_{\boldsymbol{n}}\right)$$

If row i is the sum of the two row vectors x and y

$$\begin{aligned} det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{x}+\boldsymbol{y}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) =& det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{x}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) + ...\\ & det\left(\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{y}, ..., \boldsymbol{r}_{\boldsymbol{n}}\right) \end{aligned}$$

Meaning

The determinant is a linear function of each row.

Further

Interchanging any two rows of the matrix changes the sign of the determinant

$$det\left(...,\boldsymbol{r}_i,...,\boldsymbol{r}_j,...,...\right)=det\left(...,\boldsymbol{r}_j,...,\boldsymbol{r}_i,...,...\right)$$

Finally

The determinant of any identity matrix is 1

Further

Interchanging any two rows of the matrix changes the sign of the determinant

$$det\left(...,\boldsymbol{r}_i,...,\boldsymbol{r}_j,...,...\right) = det\left(...,\boldsymbol{r}_j,...,\boldsymbol{r}_i,...,...\right)$$

Finally

The determinant of any identity matrix is 1

Outline

Cinvestav

Some Properties

Property 1

If A has a row of zeros, then $det(A) = 0$.

1. $det(A) = c \times det(A)$ for any c
2. Thus $det(A) = 0$

Some Properties

Property 1

If A has a row of zeros, then $det(A) = 0$.

Proof

1. if $A = (..., 0, ...)$, also $A = (..., c0, ...)$
2. $det\,(A) = c \times det\,(A)$ for any c
3. Thus $det\,(A) = 0$

Cinvestav

Some Properties

Property 1

If A has a row of zeros, then $det(A) = 0$.

Proof

1. if $A = (..., 0, ...)$, also $A = (..., c0, ...)$
2. $det\left(A\right) = c \times det\left(A\right)$ for any c
3. Thus $det\left(A\right) = 0$

Some Properties

Property 1

If A has a row of zeros, then $det(A) = 0$.

Proof

1. if $A = (..., 0, ...)$, also $A = (..., c0, ...)$
2. $det\,(A) = c \times det\,(A)$ for any c
3. Thus $det\,(A) = 0$

Next

Property 2

If $\boldsymbol{r}_i = \boldsymbol{r}_j$, $i \neq j$, then $det(A) = 0$.

Proof

Quite easy (Hint sing being reversed).

Next

Property 2

If $\boldsymbol{r}_i = \boldsymbol{r}_j$, $i \neq j$, then $det(A) = 0$.

Proof

Quite easy (Hint sing being reversed).

Finally

Proposition 3

If B is obtained from A by replacing $\boldsymbol{r}_i$ with $\boldsymbol{r}_i + c\boldsymbol{r}_j$, then $det(B) = det(A)$

Proof

$$
\begin{aligned}
det(B) &= det(\ldots, r_i + cr_j, \ldots, r_j, \ldots) \\
&= det(\ldots, r_i, \ldots, r_j, \ldots) + det(\ldots, cr_j, \ldots, r_j, \ldots) \\
&= det(A) + c\,det(\ldots, r_j, \ldots, r_j, \ldots) \\
&= det(A) + 0
\end{aligned}
$$

Finally

Proposition 3

If B is obtained from A by replacing $\boldsymbol{r}_i$ with $\boldsymbol{r}_i + c\boldsymbol{r}_j$, then $det(B) = det(A)$

Proof

$$
\begin{aligned}
det\left(B\right) &= det\left(\ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots\right) \\
&= det\left(\ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots\right) + det\left(\ldots, c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots\right) \\
&= det\left(A\right) + c\,det\left(\ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots\right) \\
&= det\left(A\right) + 0
\end{aligned}
$$

Finally

Proposition 3

If B is obtained from A by replacing $\boldsymbol{r}_i$ with $\boldsymbol{r}_i + c\boldsymbol{r}_j$, then $det(B) = det(A)$

Proof

$$\begin{aligned} det\left(B\right) &= det\left(..., \boldsymbol{r}_i + c\boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(..., \boldsymbol{r}_i, ..., \boldsymbol{r}_j, ...\right) + det\left(..., c\boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(A\right) + c det\left(..., \boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(A\right) + 0 \end{aligned}$$

Finally

Proposition 3

If B is obtained from A by replacing $\boldsymbol{r}_i$ with $\boldsymbol{r}_i + c\boldsymbol{r}_j$, then $det(B) = det(A)$

Proof

$$\begin{aligned} det\left(B\right) &= det\left(..., \boldsymbol{r}_i + c\boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(..., \boldsymbol{r}_i, ..., \boldsymbol{r}_j, ...\right) + det\left(..., c\boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(A\right) + c det\left(..., \boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(A\right) + 0 \end{aligned}$$

Finally

Proposition 3

If B is obtained from A by replacing $\boldsymbol{r}_i$ with $\boldsymbol{r}_i + c\boldsymbol{r}_j$, then $det(B) = det(A)$

Proof

$$\begin{aligned} det\left(B\right) &= det\left(..., \boldsymbol{r}_i + c\boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(..., \boldsymbol{r}_i, ..., \boldsymbol{r}_j, ...\right) + det\left(..., c\boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(A\right) + c det\left(..., \boldsymbol{r}_j, ..., \boldsymbol{r}_j, ...\right) \\ &= det\left(A\right) + 0 \end{aligned}$$

Outline

Therefore

Theorem

The determinant of an upper or lower triangular matrix is equal to the product of the entries on the main diagonal.

- This means all the entries beneath the main diagonal are zero.
- Using Proposition 3, we can convert it into a diagonal matrix.
- Then, by property 1
  - $det\left(A_{diag}\right) = \prod_{i=1}^{n} a_{ii} det\left(I\right) = \prod_{i=1}^{n} a_{ii}$

Therefore

Theorem

The determinant of an upper or lower triangular matrix is equal to the product of the entries on the main diagonal.

Proof

- Suppose A is upper triangular and that none of the entries on the main diagonal is 0.
- This means all the entries beneath the main diagonal are zero.
- Using Proposition 3, we can convert it into a diagonal matrix.
- Then, by property 1
  - $det\left(A_{diag}\right) = \left[\prod_i^n a_{ii}\right] det\left(I\right) = \prod_i^n a_{ii}$

Therefore

Theorem

The determinant of an upper or lower triangular matrix is equal to the product of the entries on the main diagonal.

Proof

- Suppose A is upper triangular and that none of the entries on the main diagonal is 0.
- This means all the entries beneath the main diagonal are zero.
- Using Proposition 3, we can convert it into a diagonal matrix.
- Then, by property 1
  - $det\left(A_{diag}\right) = \left[\prod_{i}^{n} a_{ii}\right] det\left(I\right) = \prod_{i}^{n} a_{ii}$

Therefore

Theorem

The determinant of an upper or lower triangular matrix is equal to the product of the entries on the main diagonal.

Proof

- Suppose A is upper triangular and that none of the entries on the main diagonal is 0.
- This means all the entries beneath the main diagonal are zero.
- Using Proposition 3, we can convert it into a diagonal matrix.
- Then, by property 1
  - $det\left(A_{diag}\right) = \left[\prod_i^n a_{ii}\right] det\left(I\right) = \prod_i^n a_{ii}$

Therefore

Theorem

The determinant of an upper or lower triangular matrix is equal to the product of the entries on the main diagonal.

Proof

- Suppose A is upper triangular and that none of the entries on the main diagonal is 0.
- This means all the entries beneath the main diagonal are zero.
- Using Proposition 3, we can convert it into a diagonal matrix.
- Then, by property 1
  - $det\left(A_{diag}\right) = \left[\prod_{i}^{n} a_{ii}\right] det\left(I\right) = \prod_{i}^{n} a_{ii}$

Remark

Question

This is the property we use to compute determinants!!! How?

Example

We have

$$\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$$

First, we have

$$r_1 = (2,1) = 2\left(2, \frac{1}{2}\right)$$

Then

$$det\left(A\right) = 2det\begin{bmatrix} 1 & \frac{1}{2} \\ 3 & -4 \end{bmatrix}$$

Cinvestav

Example

We have

$$\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$$

First, we have

$$\boldsymbol{r}_1 = (2, 1) = 2\left(2, \frac{1}{2}\right)$$

Then

$$det\left(A\right) = 2det\begin{bmatrix} 1 & \frac{1}{2} \\ 3 & -4 \end{bmatrix}$$

Example

We have

$$\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$$

First, we have

$$\boldsymbol{r}_1 = (2,1) = 2\left(2, \frac{1}{2}\right)$$

Then

$$det\left(A\right) = 2det\begin{bmatrix} 1 & \frac{1}{2} \\ 3 & -4 \end{bmatrix}$$

Cinvestav

Further

We have by proposition 3

$$det\left(A\right)=2det\begin{bmatrix}1 & \frac{1}{2}\\ 0 & -\frac{11}{2}\end{bmatrix}$$

Using Property 1

$$det\left(A\right)=2\left(-\frac{11}{2}\right)det\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}$$

Therefore

$$det\left(A\right)=-11$$

Further

We have by proposition 3

$$det\,(A) = 2det \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{11}{2} \end{bmatrix}$$

Using Property 1

$$det\,(A) = 2\left(-\frac{11}{2}\right) det \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Therefore

$$det\,(A) = -11$$

Further

We have by proposition 3

$$det\,(A) = 2det \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{11}{2} \end{bmatrix}$$

Using Property 1

$$det\,(A) = 2\left(-\frac{11}{2}\right) det \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

Therefore

$$det\,(A) = -11$$

Cinvestav

Further Properties

Property 4

The determinant of A is the same as that of its transpose A^T.

Proof

- Hint: we do an elementary row operation on A. Then, $(EA)^T = A^T E^T$

Further Properties

Property 4

The determinant of A is the same as that of its transpose A^T.

Proof

- Hint: we do an elementary row operation on A. Then, $(EA)^T = A^T E^T$

Finally

Property 5

If A and B are square matrices of the same size, then

$$det\left(AB\right) = det\left(A\right)det\left(B\right)$$

Therefore

If A is invertible:

$$\begin{aligned} det\left(AA^{-1}\right) &= det\left(A\right)det\left(A^{-1}\right) \\ &= det\left(I\right) \\ &= 1 \end{aligned}$$

Thus

$$det\left(A^{-1}\right) = \frac{1}{det\left(A\right)}$$

Finally

Property 5

If A and B are square matrices of the same size, then

$$det\left(AB\right) = det\left(A\right) det\left(B\right)$$

Therefore

If A is invertible:

$$\begin{aligned} det\left(AA^{-1}\right) &= det\left(A\right) det\left(A^{-1}\right) \\ &= det\left(I\right) \\ &= 1 \end{aligned}$$

Thus

$$det\left(A^{-1}\right) = \frac{1}{det\left(A\right)}$$

Finally

Property 5

If A and B are square matrices of the same size, then

$$det\left(AB\right) = det\left(A\right)det\left(B\right)$$

Therefore

If A is invertible:

$$\begin{aligned} det\left(AA^{-1}\right) &= det\left(A\right)det\left(A^{-1}\right) \\ &= det\left(I\right) \\ &= 1 \end{aligned}$$

Thus

$$det\left(A^{-1}\right) = \frac{1}{det\left(A\right)}$$

Finally

Definition

- If the (square) matrix A is invertible, then A is said to be non-singular.
- Otherwise, A is singular.