Mathematics for Artificial Intelligence

System of Linear Equations

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Outline

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As we saw in the previous introduction

The Development of Linear Algebra

It is as a natural extension of trying to solve systems of linear equations.

From those early attempts - Gauss and Company

- Cayley have the need to formalize fully the concept of Matrices,
  - From this simple concept a new era in Mathematics would arise.

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Example

A classic problem is to solve systems of linear equations like

$$3x + 3y = 12$$
$$x - 2y = 1$$

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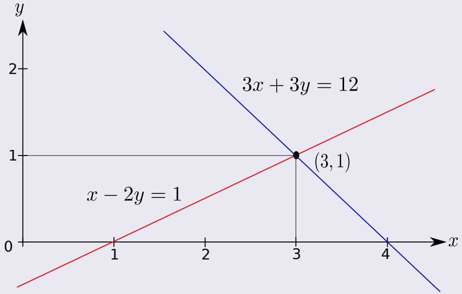

However

It is clear that once the dimension of the vector space increases beyond two

We do not have a simple geometric method to solve this problem.

We can use our knowledge of Matrices

$$\begin{pmatrix} 3 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$$

Thus, we have the equation in the following format $Ax = y$

$$x = \begin{pmatrix} x \\ y \end{pmatrix}, y = \begin{pmatrix} 12 \\ 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 3 & 3 \\ 1 & -2 \end{pmatrix}$$

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Definition of Matrices

Definition

Let K be a field, and let n, m be two integers≥ 1. An array of scalars in K:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a matrix in K. We can abbreviate the notation writing (a_{ij}) , $i = 1, ..., m$ and $j = 1, ..., n$.

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Further

We call a_{ij} the $ij-$entry of the matrix, and the i^{th} row is defined as

$$A_i = (a_{i1}, a_{i2}, ..., a_{in})$$

The j^{th} column is denoted as

$$A^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

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Addition of Matrices

Definition

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define $A + B$ be a matrix whose entry in the i^{th} row and j^{th} column is $a_{ij} + b_{ij}$.

Therefore, Is this possible?

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix} = ?$$

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The Zero Matrix

Definition

Let $A = (a_{ij})$ be q $m \times n$ matrix whose entries are all 0. This matrix is the zero matrix, $\mathbf{0}_{mn}$.

Example

Look at the Jupyter...

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Multiplication By Scalar

Definition

The multiplication by an scalar element which is defined simply as, given a matrix A and scalar c, a matrix cA whose $ij-$component is ca_{ij}.

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Remarks

It is easy to see that the set of matrices of size $m \times n$ with components in a field K form a vector space over K which can be denoted by $Mat_{m \times n}(K)$.

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Some Other Definitions

Definition

Let $A = (a_{ij})$ be an $m \times n$ matrix. The matrix $B = (b_{ij})$ such that $b_{ji} = a_{ij}$ is called **transpose** of A, and is also denoted by A^T.

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Additionally

A matrix is said to be symmetric if it is equal to its transpose i.e. if $A^T = A$.

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Matrix Multiplication

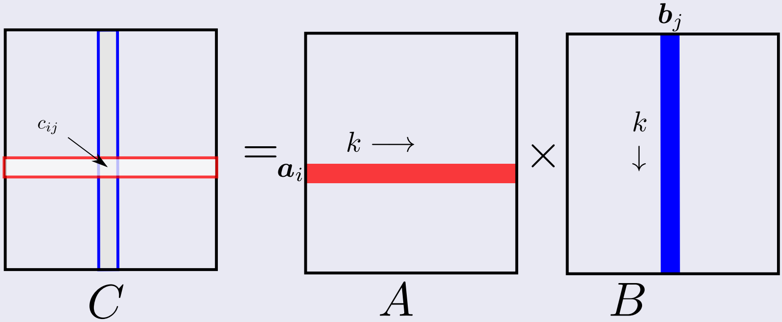
Definition

If the number of columns of A $(m \times k)$ equals the number of rows of B $(k \times n)$, then the product $C = AB$ is defined by

$$c_{ij} = \sum_{h=1}^{k} a_{ik} b_{kj} \tag{1}$$

Something Like

We have



Example

Multiply the following matrices using numpy

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix},\ B = \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 1 & 3 \end{pmatrix}$$

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A Change on Basis

We can do the following

You have a vector $\boldsymbol{x}$ in certain space V with a basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$.

Thus, we have

$$\boldsymbol{x} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n$$

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We have

Then in the basis $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$

$$[\boldsymbol{x}]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_B$$

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We need to solve the following system of equations

Solving the following system

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_n\boldsymbol{v}_n = \boldsymbol{x}$$

Or

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}_B = \boldsymbol{x}_{st}$$

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What if...?

if we have another basis for such space $A = \{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n\}$

$$
\begin{aligned}
\boldsymbol{w}_1 &= b_{11}\boldsymbol{v}_1 + b_{21}\boldsymbol{v}_2 + \cdots + b_{n1}\boldsymbol{v}_m \\
\boldsymbol{w}_2 &= b_{12}\boldsymbol{v}_1 + b_{22}\boldsymbol{v}_2 + \cdots + b_{n2}\boldsymbol{v}_m \\
\vdots &= \vdots \\
\boldsymbol{w}_n &= b_{1n}\boldsymbol{v}_1 + b_{2n}\boldsymbol{w}_2 + \cdots + b_{nn}\boldsymbol{v}_m
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Therefore, we generate the following matrix

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Therefore

We have that each column represent a vector $\boldsymbol{w}_j$ in standard basis

$$\boldsymbol{w}_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$$

We have

If we have $\boldsymbol{y} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}^T$ in the coordinates at basis $A = \{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n\}$

- Using the transition matrix idea

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

Where

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With the following property

Then using the $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ representation

$$c_i \boldsymbol{w}_i = c_i b_{1i} \boldsymbol{v}_1 + c_i b_{2i} \boldsymbol{v}_2 + \cdots + c_i b_{ni} \boldsymbol{v}_m$$

Therefore

$$\left(c_1 b_{11} + ... + c_n b_{1n}\right) \boldsymbol{v}_1 + \cdots + \left(c_1 b_{n1} + ... + c_n b_{nn}\right) \boldsymbol{v}_n = [\boldsymbol{x}]_B$$

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Or

The Coordinates for the System in basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$

Nice, What about $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\} \to \{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n\}$?

But Normally, we want to go from $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ to $\{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n\}$

Simply, given $\begin{pmatrix} d_1 & d_2 & \cdots & d_n \end{pmatrix}^T$

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Then

This is basically a way to represent the change of basis

By inner product, multiplication of matrices, inverses and matrix-vector multiplication.

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Example

We have

$$B=\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\1\\4\end{bmatrix},\begin{bmatrix}3\\2\\1\end{bmatrix}\right\},[\boldsymbol{v}]_B=\begin{bmatrix}4\\1\\-5\end{bmatrix}_B$$

Transition Matrix

$$P=\begin{bmatrix}1 & 2 & 3\\2 & -1 & 2\\-1 & 4 & 1\end{bmatrix}$$

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Therefore, we can see that

We can calculate the determinant, $det \neq 0$ linear independence

$$det \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \neq 0$$

We derive the standard coordinates of v

$$v = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}_B$$

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Now, we have that to move from one basis to another

What if we have a element in basis S

$$[\boldsymbol{v}]_S = \begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix}$$

We derive the B coordinates of vector v

$$\begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

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Solve the system or get the inverse

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Make the following example

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Change of basis from B to B'

Given an old basis B of $\mathbb{R}^n$ with transition matrix P_B

- And a new basis B' with transition matrix $P_{B'}$

How do we change from coords in the basis B to coords in the basis B'?

- Coordinates in B, then using $v = P_B [v]_B$ we change the coordinates to standard coordinates.

Then, we can do

$$[v]_{B'} = P_{B'}^{-1} v$$

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We have the following situation

$$[\boldsymbol{v}]_{B'} = P_{B'}^{-1} P_B [\boldsymbol{v}]_B$$

Then, the final transition matrix

$$M = P_{B'}^{-1} P_B = P_{B'}^{-1} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

In other words

$$M = P_{B'}^{-1} \begin{bmatrix} P_{B'}^{-1} \boldsymbol{v}_1 & P_{B'}^{-1} \boldsymbol{v}_2 & \cdots & P_{B'}^{-1} \boldsymbol{v}_n \end{bmatrix}$$

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Therefore, we have

Something Notable

- The columns of the transition matrix M from the old basis B to the new basis B'
  - They are the coordinate vectors of the old basis B with respect to the new basis B'.

Fianlly, packing everything

Theorem

- If B and B' are two bases of $\mathbb{R}^n$, with $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ then the transition matrix from B coordinates to B' coordinates is given by

$$M = \left[[\boldsymbol{v}_1]_{B'}, [\boldsymbol{v}_2]_{B'}, ..., [\boldsymbol{v}_n]_{B'}\right]$$

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Going Back to the Problem

We have

$$\begin{pmatrix} 3x + 3y \\ x - 2y \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \end{pmatrix}$$

1. The coefficients of x are in the first column.
2. The coefficients of y are in the second column.
3. The right hand side of the system is the third column.

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We have

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We have (Basically columns as place makers)

$$\begin{pmatrix} 3 & 3 & 12 \\ 1 & -2 & 1 \end{pmatrix}$$

Then, look at the following

$$\begin{pmatrix} 3 & 3 & 12 \\ 3 & -6 & 3 \end{pmatrix} : \text{Multiply the second row by 3}$$

Further

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What about?

Example

$$2x - 4y + z = 1$$
$$4x + y - z = 3$$

Then the augmented matrix

$$\begin{pmatrix} 2 & -4 & 1 & 1 \\ 4 & 1 & -1 & 3 \end{pmatrix}$$

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We have

$$\begin{pmatrix} 1 & -2 & \frac{1}{2} & \frac{1}{2} \\ 4 & 1 & -1 & 3 \end{pmatrix} : \text{ Mult row 1 by } \frac{1}{2}$$

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Further, we get a augmented matrix called an echelon form

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What operations we have been doing in the augmented matrices?

First

1. Multiply any equation by a non-zero real number (scalar).
2. Equivalent to multiplying a row of the matrix by a scalar.

Second

1. Replace any equation by the original equation plus a scalar multiple of another equation.
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Definition

These three operations are called **elementary row operations.**

Outline

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How we use only matrix operations?

Take a look at this example

$$A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 0 \end{pmatrix}$$

We have the following elementary matrix coming from the identity matrix

$$E_1 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$$

We get

$$E_1 A = \begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 2 & -1 & 0 \end{pmatrix}$$

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To add $-2\times$(row one) to row 2 in the identity matrix

$$E_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

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$$E_2 E_1 A = \begin{pmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & -\frac{11}{3} & -\frac{10}{3} \end{pmatrix}$$

Finally we can use the matrix $E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{11} \end{pmatrix}$

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1. The leading entry of every non-zero row is a 1.
2. If the leading entry of row r is in position k and the next row is not a row of zeros, then the leading entry of row $r+1$ is in position $k+j$, where $j \geq 1$.
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Examples

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This leads to

R is in reduced echelon form (Gauss-Jordan Form) if

1. R is in echelon form
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Question

Suppose A is $n \times m$ matrix. What is the maximum number of leading 1's that can appear when it's been reduced to echelon form?

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You can reduce a square matrix A into a Gauss-Jordan Form

$$E_k \cdots E_2 E_1 A = I$$

What is the name of

$$B = E_k \cdots E_2 E_1$$

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Outline

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Elementary Matrices

Definition

Elementary Matrices are constructed by performing the given row operation on the identity matrix:

1. To multiply row j of A by the scalar c use the matrix E obtained from I by multiplying j^{th} row of I by c.
2. To add $c \times row_j(A)$ to $row_k(A)$, use the identity matrix with its k^{th} row replaced by $(0, ..., 0, c, 0, ..., 0, 1, 0, ...)$.
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Elementary matrices are always square

Directly from the definition.

Elementary matrices are invertible

Basically you can revert the operations using another elementary matrix.

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Elementary row operations applied to either $A\boldsymbol{x} = \boldsymbol{y}$ or the corresponding augmented matrix $(A|\boldsymbol{y})$ do not change the set of solutions to the system.

Proof

Given the augmented matrix $(A|y)$, we multiply the by an elementary matrix E

We get

$$E(A|y) = (EA|Ey)$$

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We have this correspond to

$$EA\boldsymbol{x} = E\boldsymbol{y}$$

Now, assume that x is a solution to $Ax = y$

Then, it solves the previous system!!!

Conversely

If x solves the new system, $EAx = Ey$, multiplication by E^{-1} gives $Ax = y$

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The end result of all the row operations on $(A|\boldsymbol{y})$

$$(E_k E_{k-1} \cdots E_2 E_1 A | E_k E_{k-1} \cdots E_2 E_1 \boldsymbol{y}) = R$$

Where

R is an echelon form of $(A|y)$.

Remark

And if R is in echelon form, we can easily work out the solution.

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Outline

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Algorithm

Gauss-Jordan $(A, \boldsymbol{y})$

1. Write the augmented matrix of the system.
2. Use row operations to transform the augmented matrix in the form described below, which is called the **reduced row echelon form:**
   1. The rows (if any) consisting entirely of zeros are grouped together at the bottom of the matrix.
   2. In each row that does not consist entirely of zeros, the leftmost nonzero element is a 1 (called a leading 1 or a pivot).
   3. Each column that contains a leading 1 has zeros in all other entries.
   4. The leading 1 in any row is to the left of any leading 1's in the rows below it.
3. Stop process in step 2 if you obtain a row whose elements are all zeros except the last one on the right. In that case, the system is inconsistent and has no solutions.
4. Otherwise, finish step 2 and read the solutions of the system from the final matrix.

Example

Look at the Board for an example

Why not to try programming it!!!

Outline

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Some Definitions

Definition

- A system of equations $Ax = \boldsymbol{y}$ is consistent if there is at least one solution $\boldsymbol{x}$.
- If there is no solution, then the system is inconsistent.

Now Assume that the augmented $(A|y)$ has been reduced

To either echelon or Gauss-Jordan Form

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Gauss-Jordan Form?

Definition

The matrix A is upper triangular if any entry a_{ij} with $i > j$ satisfies $a_{ij} = 0$.

Thus

The row echelon form of the matrix is upper triangular.

Therefore

To continue the reduction to Gauss-Jordan form, it is only necessary to use each leading 1 to clean out any remaining non-zero entries in its column.

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Definition

Suppose the augmented matrix for the linear system $Ax = \boldsymbol{y}$ has been brought to echelon form.

If there is a leading 1 in any column except the last

The corresponding variable is called a leading variable.

Then

Any variable which is not a leading variable is a free variable:

$$|\text{Leading Variables}| + |\text{Free Variables}| = |\text{Number of Columns of } A|$$

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We have

If the system is consistent and there are no free variables, then the solution is unique.

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$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Second

If the system is consistent and there are one or more free variables

There are infinitely many solutions.

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Finally

The Last Case, we have a free variable, but a 1 in the last column

$$\begin{pmatrix} 1 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Outline

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We have

Definition

A homogeneous system of linear algebraic equations is one in which all the numbers on the right hand side are equal to 0:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= 0,\\
&\cdots\cdots\cdots\cdots\\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= 0.
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Remark

The homogeneous system $Ax = 0$ always has the solution $x = 0$.

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Remark

The homogeneous system $A\boldsymbol{x} = 0$ always has the solution $\boldsymbol{x} = 0$.

Thus

Non-Trivial Solutions

Any non-zero solutions to $Ax = 0$, if they exist, are called non-trivial solutions.

We can use the Gauss-Jordan Algorithm

Reducing $(A|\mathbf{0})$

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You have the following homogeneous system

$$(A|\mathbf{0}) = \begin{pmatrix} 1 & 2 & 0 & -1 & 0 \\ -2 & -2 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{pmatrix}$$

Therefore, we have after reduction

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1. Leading variables x_1, x_2
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As echelon reduced form

$$\begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, we have, using the free variable ideas

$$x_1 = 8s + 7t$$
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Therefore

We have in vector format

$$\boldsymbol{x}_H = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} | \forall s, t \in \right\}$$

Outline

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We have

Basic Properties for a $A_{m\times n}$

1. The number of leading variables is $\leq \min(m,n)$
2. The number of non-zero equations in the echelon form of the system is equal to the number of leading entries.
3. The number of free variables plus the number of leading variables $= n$, the number of columns of A.
4. The homogeneous system $Ax = 0$ has non-trivial solutions if and only if there are free variables.
5. A homogeneous system of equations is always consistent (At least a solution)

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We have

Theorem

If x is a solution to $Ax = 0$, then so is cx for any real number c.

Proof: Quite simple

Theorem

If x and y are two solutions to the homogeneous equation, then so is $x + y$.

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It is also simple!!!

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Superposition Principle

We have

These two properties constitute the famous principle of superposition which holds for homogeneous systems.

Restating

If x and y are two solutions to the homogeneous equation $Ax = 0$, then any linear combination of x and y is also a solution.

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Superposition Principle

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These two properties constitute the famous principle of superposition which holds for homogeneous systems.

Restating

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two solutions to the homogeneous equation $A\boldsymbol{x} = 0$, then any linear combination of $\boldsymbol{x}$ and $\boldsymbol{y}$ is also a solution.

Outline

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We have

Definition

The system $A\boldsymbol{x} = \boldsymbol{y}$ is inhomogeneous if it is not homogeneous.

Example

$$
\begin{aligned}
x_1 + 2x_2 - x_4 &= 1 \\
-2x_1 - 3x_2 + 4x_3 + 5x_4 &= 2 \\
2x_1 + 4x_2 - 2x_4 &= 3
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Augmented Matrix

$$(A|y) = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 3 \end{pmatrix}$$

The row echelon form of the augmented matrix is

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The reduced echelon form is

$$\begin{pmatrix} 1 & 0 & -8 & -7 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem, the third equation now reads

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

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Reasoning

If the original system has a solution

Then performing elementary row operations will give us an equivalent system with the same solution.

But this equivalent system of equations is inconsistent

So the original system is also inconsistent.

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In General

If the echelon form of $(A|y)$ has a leading 1 in any position of the last column, the system of equations is inconsistent.

Then

It is not true that any inhomogeneous system with the same matrix A is inconsistent

It depends completely on the particular $\boldsymbol{y}$ which sits on the right hand side

Are all the inhomogeneous matrices inconsistent?

Nope

$$(A|y) = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 2 \end{pmatrix}$$

After echelon form

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Or

We have

$$\boldsymbol{x}_I = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -7 \\ 4 \\ 0 \\ 0 \end{pmatrix} | \forall s, t \in \right\}$$

Thus, by fixing

For example, $s = t = 0$

$$\boldsymbol{x}_p = \begin{pmatrix} -7 \\ 4 \\ 0 \\ 0 \end{pmatrix}$$

The general solution to the inhomogeneous system

$$\boldsymbol{x}_I = \boldsymbol{x}_H + \boldsymbol{x}_p$$

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The general solution to the inhomogeneous system

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Finally

Theorem

- Let $\boldsymbol{x}_p^1$ and $\boldsymbol{x}_p^2$ be two solutions to $Ax = y$. Then their difference $\boldsymbol{x}_p^1 - \boldsymbol{x}_p^2$ is a solution to the homogeneous equation $A\boldsymbol{x} = 0$.
- The general solution to $A\boldsymbol{x} = \boldsymbol{y}$ can be written as $\boldsymbol{x}_I = \boldsymbol{x}_H + \boldsymbol{x}_p$ where $\boldsymbol{x}_H$ denotes the general solution to the homogeneous system.