Mathematics for Artificial Intelligence

Vector Spaces

Andres Mendez-Vazquez

March 14, 2020

Outline

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Introduction

What is this class about?

It is clear that the use of mathematics is essential for the data mining and machine learning fields.

Therefore...

The understanding of Mathematical Modeling is part of the deal...

If you want to be

A Good Data Scientist!!!

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Example

Imagine

A web surfer moves from a web page to another web page...

- Question: How do you model this?

You can use a graph!!!

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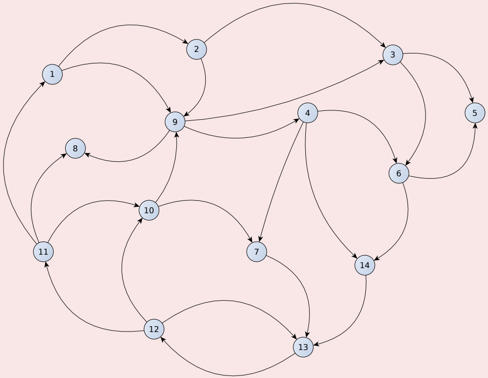
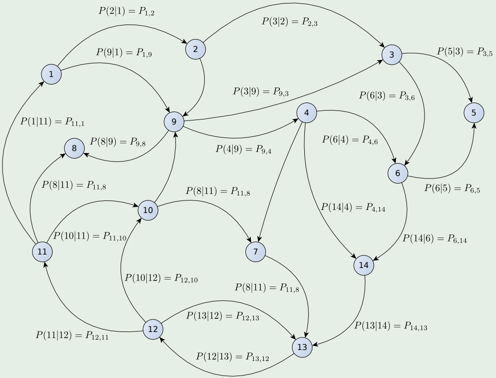


Now

Add Some Probabilities



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Thus

We can build a matrix

$$M = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{pmatrix} \quad (1)$$

Thus, it is possible to obtain certain information by looking at the eigenvector and eigenvalues

These vectors $v_\lambda' s$ and values $\lambda' s$ have the property that

$$Mv_\lambda = \lambda v_\lambda \quad (2)$$

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This is the Basis of Page Rank in Google

For example

- Look at this example...

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About 4000 years ago

Babylonians knew how to solve the following kind of systems

$$ax + by = c$$
$$dx + ey = f$$

As always the first steps in any field of knowledge tend to be slow

It is only after the death of Plato and Aristotle, that the Chinese (Nine Chapters of the Mathematical Art 200 B.C.) were able to solve 3×3 system.

By working an "elimination method"

Similar to the one devised by Gauss 2000 years later for general systems.

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Not only that

The Matrix

Gauss defined implicitly the concept of a Matrix as linear transformations in his book "Disquisitions."

The Final Definition of Matrix

It was introduced by Cayley in two papers in 1850 and 1858 respectively, which allowed him to prove the important Cayley-Hamilton Theorem.

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They are important for many calculations as

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\dots\dots\dots\dots\dots\dots\dots\dots\\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n =& b_2.
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It is clear

We would like to collect those linear equations in a compact structure that allows for simpler manipulation.

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Therefore, we have

For example

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ and } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Using a little of notation

$$Ax = b$$

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$$A\boldsymbol{x} = \boldsymbol{b}$$

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As always, we star with a simple fact

Everything is an element in a set.

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For example

- The set of Real Numbers $\mathbb{R}$.
- The set of n-tuples in $\mathbb{R}^n$.
- The set of Complex Number $\mathbb{C}$.

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Definition

We shall say that K is a field if it satisfies the following conditions for the addition

Property	Formalism
Addition is Commutative	$x + y = y + x$ for all $x, y \in K$
Addition is associative	$x + (y + z) = (x + y) + z$ for all $x, y, z \in K$
Existence of 0	$x + 0 = x$, for every $x \in K$
Existence of the inverse	$\forall x$ there is $\exists -x \implies x + (-x) = 0$

Furthermore

We have the following properties for the product

Property	Formalism
Product is Commutative	$xy = yx$ for all $x, y \in K$
Product is associative	$x(yz) = (xy)z$ for all $x, y, z \in K$
Existence of 1	$1x = x1 = x$, for every $x \in K$.
Existence of the inverse	x^{-1} or $\frac{1}{x}$ in K such that $xx^{-1} = 1$.
Multiplication is Distributive over addition	$x(y+z) = xy + xz$, for all $x, y, z \in K$

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Examples

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Then, we get a crazy moment

How do we relate these numbers to obtain certain properties

- We have then the vector and matrix structures for this...

$$\begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \text{ and } \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$$

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Vector Space V

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Properties

We have then

1. Given elements $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ of V, we have $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$.
2. There is an element of V, denoted by O, such that $O+\boldsymbol{u}=\boldsymbol{u}+O=\boldsymbol{u}$ for all elements $\boldsymbol{u}$ of V.
3. Given an element $\boldsymbol{u}$ of V, there exists an element $-\boldsymbol{u}$ in V such that $\boldsymbol{u}+(-\boldsymbol{u})=O$.
4. For all elements $\boldsymbol{u}, \boldsymbol{v}$ of V, we have $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$.
5. For all elements $\boldsymbol{u}$ of V, we have $1 \cdot \boldsymbol{u}=\boldsymbol{u}$.
6. If c is a number, then $c(\boldsymbol{u}+\boldsymbol{v})=c\boldsymbol{u}+c\boldsymbol{v}$.
7. if a, b are two numbers, then $(ab)\boldsymbol{v}=a(b\boldsymbol{v})$.
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Notation

First, $\boldsymbol{u} + (-\boldsymbol{v})$

As $\boldsymbol{u} - \boldsymbol{v}$.

For O

We will write sometimes 0.

The elements in the field K

They can receive the name of number or scalar.

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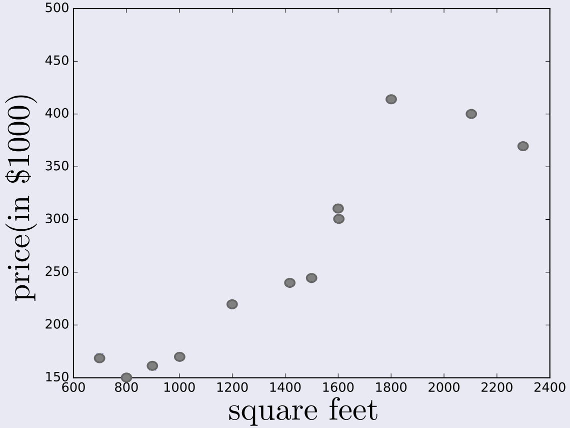
Many Times

We have this kind of data sets (House Prices)

$$\begin{pmatrix} \text{Squared Feet} \\ \text{Price} \end{pmatrix} \rightarrow \begin{pmatrix} 2104 \\ 400 \end{pmatrix} \begin{pmatrix} 1800 \\ 460 \end{pmatrix} \begin{pmatrix} 1600 \\ 300 \end{pmatrix} \begin{pmatrix} 2300 \\ 370 \end{pmatrix} \cdots$$



Therefore

We can represent these relations as vectors

$$\begin{pmatrix} \text{Squared Feet} \\ \text{Price} \end{pmatrix} = \left\{ \begin{pmatrix} 2104 \\ 400 \end{pmatrix}, \begin{pmatrix} 1800 \\ 460 \end{pmatrix}, \begin{pmatrix} 1600 \\ 300 \end{pmatrix}, \dots \right\}$$

Thus, we can start using

- All the tools that Linear Algebra can provide!!!

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We can adjust a line/hyper-plane to be able to forecast prices

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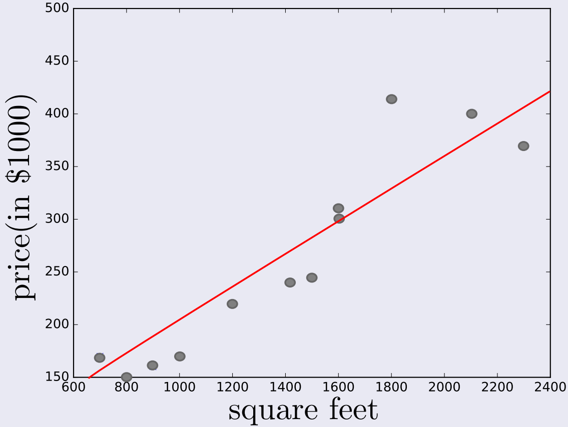

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To find such hyper-plane

- To do forecasting on the prices of a house given its surface size!!!

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Definition

Let V a vector space and $W \subseteq V$, thus W is a **subspace** if:

1. If $v, w \in W$, then $v + w \in W$.
2. If $v \in W$ and $c \in K$, then $cv \in W$.
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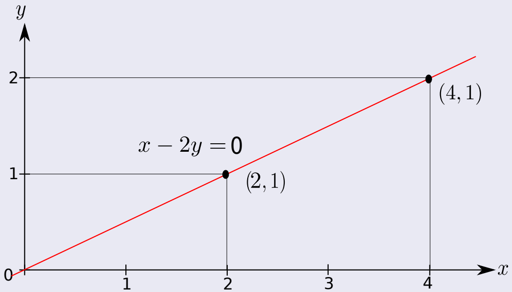
Some ways of recognizing Sub-spaces

Theorem

A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\boldsymbol{v}, \boldsymbol{w} \in W$ and each scalar $c \in K$ the vector $c\boldsymbol{v} + \boldsymbol{w} \in W$.

Example

For $\mathbb{R}^2$



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Linear Combinations

Definition

Let V an arbitrary vector space, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n \in V$ and $x_1, x_2, ..., x_n \in K$. Then, an expression like

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + ... + x_n\boldsymbol{v}_n \tag{3}$$

is called a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$.

Classic Examples

Endmember Representation in Hyperspectral Images

Look at the board

Geometric Representation of addition of forces in Physics

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Properties and Definitions

Theorem

Let V be a vector space over the field K. The intersection of any collection of sub-spaces of V is a subspace of V.

- The sub-space spanned by S is defined as the intersection W of all sub-spaces of V which contains S.
- When S is a finite set of vectors, $S = \{v_1, v_2, ..., v_n\}$, we shall simply call W the sub-space spanned by the vectors $v_1, v_2, ..., v_n$.

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- The **sub-space spanned** by S is defined as the intersection W of all sub-spaces of V which contains S.
- When S is a finite set of vectors, $S = \{v_1, v_2, \ldots, v_n\}$, we shall simply call W the **sub-space spanned by the vectors** $v_1, v_2, \ldots, v_n$.

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Properties and Definitions

Theorem

Let V be a vector space over the field K. The intersection of any collection of sub-spaces of V is a subspace of V.

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We get the following Theorem

Theorem

The subspace spanned by $S \neq \emptyset$ is the set of all linear combinations of vectors in S.

Outline

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Linear Independence

Definition

Let V be a vector space over a field K, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n \in V$. We have that $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are linearly dependent over K if there are elements $a_1, a_2, ..., a_n \in K$ not all equal to 0 such that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + ... + a_n\boldsymbol{v}_n = O$$

Thus

Therefore, if there are not such numbers, then we say that $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are linearly independent.

We have the following

Example!!!

Linear Independence

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Example!!!

Basis

Definition

If elements $v_1, v_2, ..., v_n$ generate V and in addition are linearly independent, then $\{v_1, v_2, ..., v_n\}$ is called a **basis** of V. In other words the elements $v_1, v_2, ..., v_n$ form a basis of V.

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Examples

The Classic Ones!!!

Outline

Coordinates

Theorem

Let V be a vector space. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ be linearly independent elements of V. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be numbers. Suppose that we have

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2 + \cdots + y_n\boldsymbol{v}_n \quad (4)$$

Then, $x_i = y_i$ for all $i = 1, \ldots, n$.

Coordinates

Let V be a vector space, and let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ be a basis of V

For all $\boldsymbol{v} \in V$, $\boldsymbol{v} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n$.

Thus, this n-tuple is uniquely determined by v

We will call $(x_1, x_2, \ldots, x_n)$ as the coordinates of v with respect to the basis.

The $n-$tuple $X = (x_1, x_2, \ldots, x_n)$

It is the **coordinate vector** of v with respect to the basis $\{v_1, v_2, ..., v_n\}$.

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Outline

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Properties of a Basis

Theorem - (Limit in the size of the basis)

Let V be a vector space over a field K with a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\}$. Let $\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n$ be elements of V, and assume that $n > m$. Then $\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n$ are linearly dependent.

Examples

- Matrix Space
- Canonical Space vectors
- etc

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Some Basic Definitions

We will define the dimension of a vector space V over K

As the number of elements in the basis.

- Denoted by $\dim_K V$, or simply $\dim V$

Therefore

A vector space with a basis consisting of a finite number of elements, or the zero vector space, is called a **finite dimensional**.

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Is this number unique?

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Maximal Set of Linearly Independent Elements

Theorem

Let V be a vector space, and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ a maximal set of linearly independent elements of V. Then, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ is a basis of V.

Theorem

Let V be a vector space of dimension n, and let $v_1, v_2, ..., v_n$ be linearly independent elements of V. Then, $v_1, v_2, ..., v_n$ constitutes a basis of V.

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Equality between Basis

Corollary

Let V be a vector space and let W be a subspace. If $\dim W = \dim V$ then $V = W$.

Proof

At the Board...

Corollary

Let V be a vector space of dimension n. Let r be a positive integer with $r < n$, and let $v_1, v_2, ..., v_r$ be linearly independent elements of V. Then one can find elements $v_{r+1}, v_{r+2}, ..., v_n$ such that $\{v_1, v_2, ..., v_n\}$ is a basis of V.

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At the Board...

Finally

Theorem

Let V be a vector space having a basis consisting of n elements. Let W be a subspace which does not consist of O alone. Then W has a basis, and the dimension of W is $\leq n$.

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Outline

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Feature Vector

Definition

A **feature vector** is a n-dimensional vector of numerical features that represent an object.

Why is this important?

This allows to use linear algebra to represent basic classification algorithms because

- The tuples $\{(x,y) | x \in K^n, y \in K\}$ can be easily used to design specific algorithms.

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Outline

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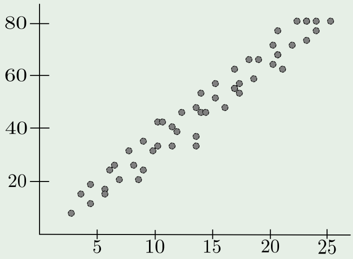
Least Squared Error

We need to fit a series of points against a certain function



We want

The general problem is given a set of functions $f_1, f_2, ..., f_K$ find values of coefficients $a_1, a_2, ..., a_k$ such that the linear combination:

$$y = a_1 f_1(x) + \cdots + a_K f_K(x) \quad (5)$$

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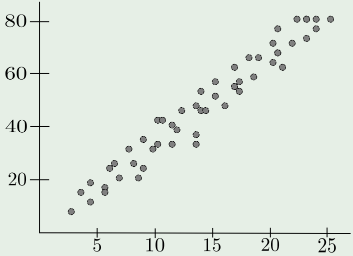

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Thus

We have that given the datasets $\{(\boldsymbol{x}_1, y_1), ..., (\boldsymbol{x}_N, y_N)\}$

$$\overline{\boldsymbol{x}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{x}_i. \tag{6}$$

Thus, we have the following problem

A Possible High Variance on the Data itself

Variance

$$\sigma_{\boldsymbol{x}}^2 = \frac{1}{N}\sum_{i=1}^{N} (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) \tag{7}$$

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Assume

A linear equation $y = ax + b$, then $y - (ax + b) \approx 0$.

We get a series of errors given the following observations $\{(x_1, y_1), \ldots, (x_N, y_N)\}$

$$\{y_1 - (ax_1 + b), \ldots, y_N - (ax_N + b)\}.$$

Then, the mean should be really small (If it is a good fit)

$$\sigma^2_{y-(ax+b)} = \frac{1}{N} \sum_{i=1}^{N} (y_i - (ax_i + b))^2 \quad (8)$$

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We can define the following error $E_i\left(a,b\right) = y - \left(ax+b\right)$

$$E\left(a,b\right) = \sum_{i=1}^{N} E_i\left(a,b\right) = \sum_{i=1}^{N} \left(y_i - \left(ax_i + b\right)\right) \quad (9)$$

We want to minimize the previous equation

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Finally

Look at the Board

We need to obtain the necessary equations.