Analysis of Algorithms

Complexity and Sorting

Andres Mendez-Vazquez

September 17, 2020

Outline

Outline

1. Algorithmic Complexity Analysis
   - Introduction
   - Asymptotic Bounds
   - Correctness of Algorithms
   - Excercises

Algorithm efficiency

When measuring an algorithm efficiency we must consider:

- Speed.
- Memory usage.
- Scalability.

Algorithm efficiency

When measuring an algorithm efficiency we must consider:

- Speed.
- Memory usage.
- Scalability.

Measuring speed!

- Speed is measured in terms of the number of operations relative to the size of the input.

Algorithm efficiency

When measuring an algorithm efficiency we must consider:

- Speed.
- Memory usage.
- Scalability.

Measuring speed!

- Speed is measured in terms of the number of operations relative to the size of the input.

Algorithm efficiency

When measuring an algorithm efficiency we must consider:

- Speed.
- Memory usage.
- Scalability.

Measuring speed!

- Speed is measured in terms of the number of operations relative to the size of the input.

Asymptotic bounds

Intuition

An asymptotic bound is a curve that represents the limit of a function.

Asymptotic bounds

Intuition

An asymptotic bound is a curve that represents the limit of a function.

For the purpose of analyzing the speed of an algorithm, tree typical asymptotic bounds are used.

1. Big O (Upper bound)
2. Big Ω(Lower bound)
3. Big Θ(Expected bound)

Asymptotic bounds

Intuition

An asymptotic bound is a curve that represents the limit of a function.

For the purpose of analyzing the speed of an algorithm, tree typical asymptotic bounds are used.

1. Big O (Upper bound)
2. Big Ω(Lower bound)
3. Big Θ(Expected bound)

Asymptotic bounds

Intuition

An asymptotic bound is a curve that represents the limit of a function.

For the purpose of analyzing the speed of an algorithm, tree typical asymptotic bounds are used.

1. Big O (Upper bound)
2. Big Ω(Lower bound)
3. Big Θ(Expected bound)

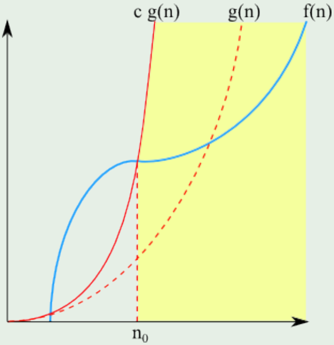
Outline

1. Algorithmic Complexity Analysis
   - Introduction
   - Asymptotic Bounds
   - Correctness of Algorithms
   - Excercises

Big O (Upper bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in O(g(n))$.



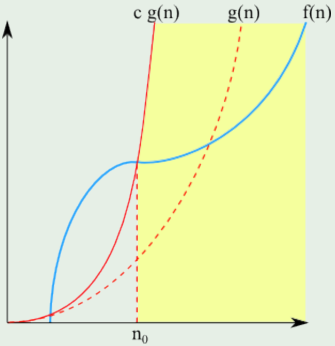
Definition

- $f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Big O (Upper bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in O(g(n))$.



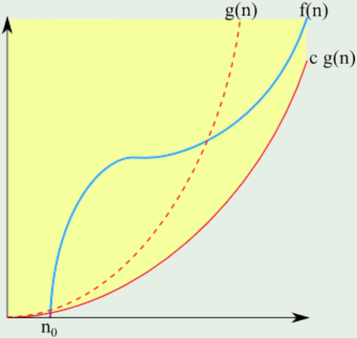
Definition

- $f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Big Ω (Lower bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in \Omega(g(n))$.



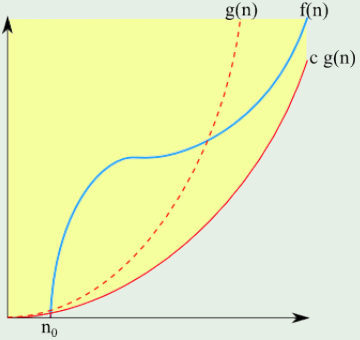
Definition

- $f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

Big Ω (Lower bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in \Omega(g(n))$.



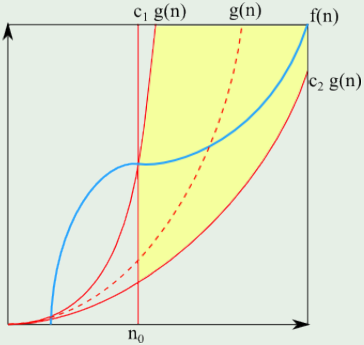
Definition

- $f(n) \in \Omega(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$.

Big Θ (Expected bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in \Theta(g(n))$.



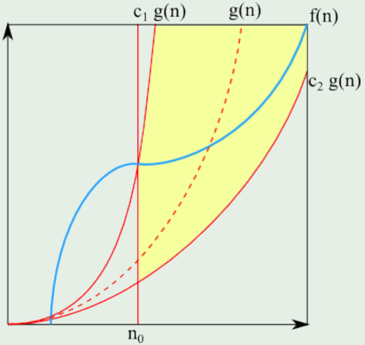
Definition

- $f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Big Θ (Expected bound)

Intuition

- Let $f(n)$ and $g(n)$ be two real valued functions, lets build intuition on the meaning of $f(n) \in \Theta(g(n))$.



Definition

- $f(n) \in O(g(n))$ if there exists $c, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

Outline

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Correctness of an algorithm

Loop invariant and loop conditional

- A loop invariant is a condition that is necessarily true immediately before and immediately after each iteration of a loop.
- A loop conditional is a statements that controls the termination of the loop.
- Both loop invariant and loop conditional must be different conditions.

Facts!

- To prove an algorithm is correct we must find a loop conditional that ensures the algorithm terminates.
- The loop invariant mus be true:
  - Before the loop starts
  - Before each iteration of the loop
  - After the loop terminates

Outline

Well, now you know the basics. Time to work!

From Dasgupta's Algorithms book, exercise 0.1

Using the definition in each of the following situations indicate wether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$).

1. $f(n) = n - 100$, $g(n) = n - 200$
2. $f(n) = n2^n$, $g(n) = 3^n$

Well, now you know the basics. Time to work!

From Dasgupta's Algorithms book, exercise 0.1

Using the definition in each of the following situations indicate wether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$).

1. $f(n) = n - 100$, $g(n) = n - 200$
2. $f(n) = n2^n$, $g(n) = 3^n$

Well, now you know the basics. Time to work!

From Dasgupta's Algorithms book, exercise 0.1

Using the definition in each of the following situations indicate wether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$).

1. $f(n) = n - 100$, $g(n) = n - 200$
2. $f(n) = n2^n$, $g(n) = 3^n$

Exercise

Let's try this one!

Show that $\sum_{k=1}^{n} \frac{1}{k^2}$ is bounded by a constant. (help me here!).

Exercise

From Cormen's book exercise 2.3-4

We can express insertion sort as a recursive procedure as follows. In order to sort $A[1...n]$, we recursively sort $A[1...n-1]$ and then insert $A[n]$ into the sorted array. Write a recurrence for the running time of this recursive version of insertion sort.

Exercise

From Cormen's book exercise 3.1-7

Prove that $o(g(n)) \cap \omega(g(n))$ is the empty set.

Proof

Remember o

$$o(g(n)) = \{f(n) |\ \text{For any } c > 0 \text{ there exists } n_0 > 0 \text{ s.t. } 0 \leq f(n) < cg(n)\ \forall n \geq n_0\}$$

Remember ω

$$\omega(g(n)) = \{f(n) |\ \text{For any } c > 0 \text{ there exists } n_0 > 0 \text{ s.t. } 0 \leq cg(n) < f(n)\ \forall n \geq n_0\}$$

Proof

Remember o

$$o(g(n)) = \{f(n) \mid \text{For any } c > 0 \text{ there exists } n_0 > 0 \text{ s.t. } 0 \leq f(n) < cg(n) \ \forall n \geq n_0\}$$

Remember ω

$$\omega(g(n)) = \{f(n) \mid \text{For any } c > 0 \text{ there exists } n_0 > 0 \text{ s.t. } 0 \leq cg(n) < f(n) \ \forall n \geq n_0\}$$

Exercise

From Cormen's book exercise 3.2-8

Show that $k \ln k = \Theta(n)$ implies that $k = \Theta\left(\frac{n}{\ln n}\right)$.

Proof

By definition

- Exist c_1, c_2 and n_0 such that for a specific k

$$c_1 n \leq k \ln k \leq c_2 n, \;\; n > n_0$$

Then we can choose

$$k \leq n_0$$

Therefore

$$\log k \leq \log n \Rightarrow \frac{1}{\log k} > \frac{1}{\log n}$$

Proof

By definition

- Exist c_1, c_2 and n_0 such that for a specific k

$$c_1 n \leq k \ln k \leq c_2 n, \;\; n > n_0$$

Then we can choose

$$k \leq n_0$$

Therefore

$$\log k \leq \log n \Rightarrow \frac{1}{\log k} \geq \frac{1}{\log n}$$

Proof

By definition

- Exist c_1, c_2 and n_0 such that for a specific k

$$c_1 n \leq k \ln k \leq c_2 n, \;\; n > n_0$$

Then we can choose

$$k \leq n_0$$

Therefore

$$\log k \leq \log n \Rightarrow \frac{1}{\log k} > \frac{1}{\log n}$$

Exercise

From Cormen's book exercise 4.3-1, 4.3-6

1. Show that the solution of $T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1$ is $O(\lg n)$.
2. Show that the solution of $T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$ is $\Omega(n \lg n)$.

Exercise

From Cormen's book exercise 4.3-1, 4.3-6

1. Show that the solution of $T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1$ is $O(\lg n)$.
2. Show that the solution of $T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$ is $\Omega(n \lg n)$.

Proof

By susbstitution method $T(n) \leq c \lg n$

$$
\begin{aligned}
T(n) \leq & 2c \lg \left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 1 \\
\leq & 2c \left(\lg n - \lg 2 \right) + 1 \\
\leq & 2c \lg n - 2c + 1
\end{aligned}
$$

Thus, we need to have

$$-2c + 1 < 0 \to c > \frac{1}{2}$$

Proof

By susbstitution method $T(n) \leq c \lg n$

$$
\begin{aligned}
T(n) \leq & 2c \lg\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1 \\
\leq & 2c\left(\lg n - \lg 2\right) + 1 \\
\leq & 2c \lg n - 2c + 1
\end{aligned}
$$

Thus, we need to have

$$
-2c + 1 < 0 \rightarrow c > \frac{1}{2}
$$

Exercise

From Cormen's book exercise 2.1-3

Consider the searching problem:

Input: A sequence of n numbers $A = \langle a_1, a_2, ..., a_n \rangle$ and a value v.

Output: An index i such that $v = A[i]$ or the special value NIL if v does not appear in A.

Write pseudocode for linear search, which scans through the sequence, looking for v. Using a loop invariant, prove that your algorithm is correct. Make sure that your loop invariant fulfills the three necessary properties.