Analysis of Algorithms

NP-Completeness

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Outline

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Polynomial Time

Algorithms Until Now

All the algorithms, we have studied this far have been polynomial-time algorithms.

However

- There is a collection of algorithms that cannot being solved in polynomial time!!!

Example

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The Intuition

Class P

They are algorithms that on inputs of size *n* have a worst case running time of $O\left(n^k\right)$ for some constant k.

Class NP

Informally, the Non-Polynomial (NP) time algorithms are the ones that cannot be solved in $O\left(n^k\right)$ for any constant k.

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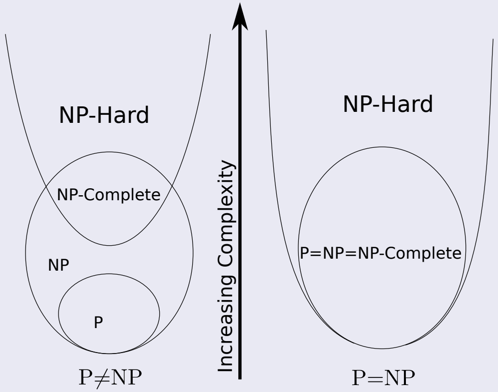
The Two Views of The World

The Paradox



However, There are differences pointing to $P \neq NP$

Shortest Path is in P

Even with negative edge weights, we can find a shortest path for a single source in a directed graph $G = (V, E)$ in $O(VE)$ time.

Longest Path is in NP

Merely determining if a graph contains a simple path with at least a given number of edges is NP.

It is more

A simple change on a polynomial time problem can move it from P to NP.

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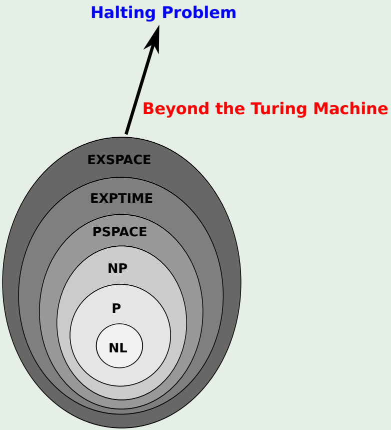
And here, a simplified classification of problems

Different Complexity Classes

Complexity Class	Model of Computation	Resource Constraint
P	Deterministic Turing Machine	Solvable using poly(n) time
NP	Non-deterministic Turing Machine	Verifiable in poly(n) time
PSPACE	Deterministic Turing Machine	Solvable using poly(n) Space
EXPTIME	Deterministic Turing Machine	Solvable using $2^{\text{poly}(n)}$ time
EXPSPACE	Deterministic Turing Machine	Space $2^{\text{poly}(n)}$
NL	Non-deterministic Turing Machine	Space $O(\log n)$

Graphically

What is contained into what



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We start by formalizing the notion of polynomial time

Polynomial Time Problems

We generally regard these problems as tractable, but for philosophical, not mathematical, reasons.

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It is possible to regard a problem with complexity $O(n^{100})$ as intractable, really few practical problems require time complexities with such high degree polynomial.

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Experience has shown that once the first polynomial-time algorithm for a problem has been discovered, more efficient algorithms often follow.

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Reasons

Second

For many reasonable models of computation, a problem that can be solved in polynomial time in one model can be solved in polynomial time in another.

Example

Problems that can be solved in polynomial time by a serial random-access machine can be solved in a Turing Machine.

- Since polynomials are closed under addition, multiplication, and composition.

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Reasons

Why?

For example, if the output of one polynomial time algorithm is fed into the input of another, the composite algorithm is polynomial.

Polynomial time

To understand a polynomial time we need to define:

- What is the meaning of an abstract problem?
- How to encode problems.
- A formal language framework.

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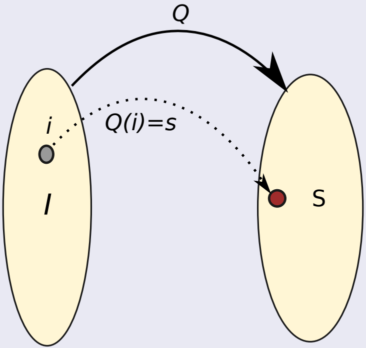
What do we need to understand the polynomial time?

What is an abstract problem?

We define an **abstract** problem Q to be a binary relation on a set I of problem **instances** and a set S of problem solutions:

$$Q : I \rightarrow S,\ Q(i) = s \tag{1}$$



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We can view an abstract decision problem as a function that maps the instance set I to the solution set $\{0,1\}$:

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Example of optimization problem: SHORTEST-PATH

The problem SHORTEST-PATH is the one that associates each graph G and two vertices with the shortest path between them.

Problem, this is a optimization problem

We need a decision problem!!!

What do we do?

We can cast a given optimization problem as a related decision problem by imposing a bound on the value to be optimized.

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Example PATH Problem

Given a undirected graph G, vertices u and v, and an integer k, we need to answer the following question:

- **Does a path exist from u to v consisting of at most k edges?**

In a more formal way

We have the following optimization problem

$$
\begin{aligned}
\min_{t} \;\; & d\left[t\right] \\
& s.t. d[v] \leq d[u] + w(u,v) \text{ for each edge } (u,v) \in E \\
& d\left[s\right] = 0
\end{aligned}
$$

Then, we have the following decision problem

$PATH = \{\langle G, u, v, k\rangle \,|\, G = (V, E)$ is an undirected graph,
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Given a graph $G = (V, E)$:

- We can encode each vertex $\{1, 2, ...\}$ as $\{0, 1, 10, ...\}$
- Then, each edge, for example $\{1, 2\}$ as $\{0, 1\}$
- Clearly you need to encode some kind of delimiter for each element in the description

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When it is provided with a problem instance i of length $|i| = n$.

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Using Encodings

Something Notable

Given an abstract decision problem Q mapping an instance set I to $\{0, 1\}$.

Then

An encoding $e : I \longrightarrow \{0, 1\}^*$ can induce a related concrete decision problem, denoted as by $e(Q)$.

- Then the solution to the concrete problem instance $e(i) \in \{0, 1\}^*$ is also $Q(i)$.

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- **HOWEVER, it depends quite heavily on the encoding.**

An example of a really BAD situation

Imagine the following

- You could have an algorithm that takes k as the sole input with an algorithm that runs in $\Theta(k)$.

Now, if the integer is provided in an unary representation (Only ones)

Quite naive!!!

Then

- Running time of the algorithm is $O(n)$ on n-length inputs, which is polynomial.

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More Observations

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We cannot talk about solving an abstract problem without specifying the encoding!!!

Nevertheless

If we rule out expensive encodings such as unary ones, the actual encoding of a problem makes little difference to whether the problem can be solved in polynomial time.

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Some properties of the polynomial encoding

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For some set I of problem instances, we say that two encodings e_1 and e_2 are polynomially related

- if there exist two polynomial time computable functions f_{12} and f_{21} such that for any $i \in I$, we have $f_{12}(e_1(i)) = e_2(i)$ and $f_{21}(e_2(i)) = e_1(i)$.

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An important lemma

Lemma 34.1

Let Q be an abstract decision problem on an instance set I, and let e_1 and e_2 be polynomially related encodings on I. Then, $e_1(Q) \in P$ if and only if $e_2(Q) \in P$.

Proof in the board

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Definitions

1. An alphabet Σ is a finite set of symbols.
2. A language L over Σ is any set of strings made up of symbols from Σ^*.
3. The empty language is ε.
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Union, intersection and complement

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- $L_1 \bigcup L_2 = \{x \in \Sigma^* | \ x \in L_1 \vee x \in L_2\}$
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This allow us to define a language that is solvable by Q

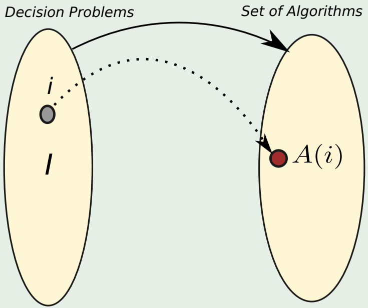
We can write it down as the language

$$L = \{x \in \Sigma^* | Q(x) = 1\} \quad (2)$$

Thus, we can express the duality Decision Problem-Algorithm

Important

The formal-language framework allows to express concisely the relation between decision problems and algorithms that solve them.



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- An algorithm A **accepts** a string $x \in \{0,1\}^*$, if given x, the algorithm's output is $A(x) = 1$.

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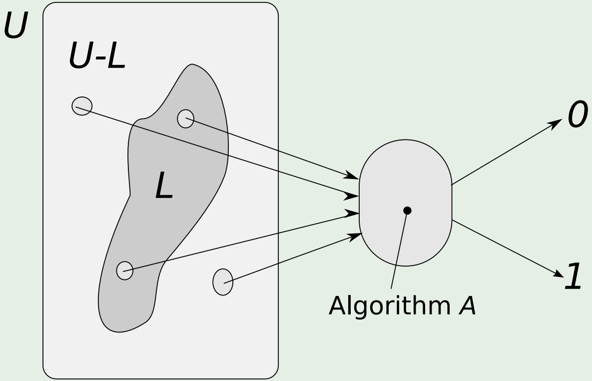
We need to be more stringent

A language L is decided by an algorithm A if every binary string in L is accepted by A and every binary string not in L is rejected by A.



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Given a string $x \in \{0,1\}^*$, only one of two things can happen:

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Now, it is possible to define acceptance in polynomial time

We have that

- A language L is **accepted** in polynomial time by an algorithm A, if it is accepted by A in polynomial time:
  - There exists a constant k such that for any n-length string $x \in L \Rightarrow$ algorithm A accepts x in time $O\left(n^k\right)$.

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Example of polynomial accepted problems

Example of polynomial accepted problem

$$PATH = \{\langle G, u, v, k\rangle \,|G = (V, E) \text{ is an undirected graph,}$$
$$u, v \in V, k \geq 0 \text{ is an integer and there exist a path from}$$
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1. Suppose that given a program p and an input x.
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Then

We have the following program

function PARADOX$(z : file)$

1. if TERMINATES(z, z) goto 1

Notice what paradox does

It terminates if and only if program z does not terminate when given its own code as input.

1. Case I : The PARADOX terminates $\Rightarrow$ Then TERMINATE says false!!!
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Complexity Classes

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We can informally define a **complexity class** as a set of languages.

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The membership to this class is determined by a **complexity measure**, such as running time, of an algorithm that determines whether a given string x belongs to language L.

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We can use this framework to say the following

- $P = \{L \subseteq \{0,1\}^* \mid$ There exists an algorithm A that decides L in polynomial time$\}$
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Theorem 34.2

$P = \{L \mid L$ is accepted by a polynomial-time algorithm$\}$

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Exercises

From Cormen's book solve

- 34.1-1
- 34.1-2
- 34.1-3
- 34.1-4
- 34.1-5
- 34.1-6

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What is verification?

Intuitive definition

Given an instance of a decision problem:

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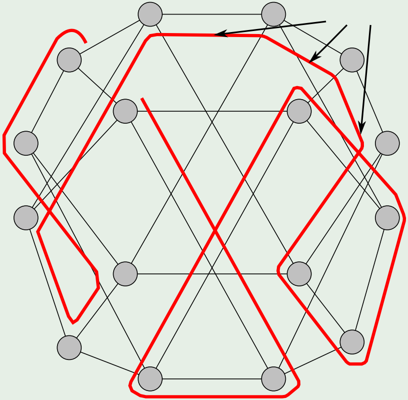
Example of verifiable problems

Hamiltonian cycle

A Hamiltonian cycle of an undirected graph $G = (V, E)$ is a simple cycle that contains each vertex in V.



As a formal language

Does a graph G have a Hamiltonian cycle?

$$HAM-CYCLES = \{\langle G\rangle \,|G \text{ is a Hamiltonian graph}\} \quad (4)$$

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Decision algorithm for Hamiltonian

Given an instance $< G >$ and encode it

- If we use the "reasonable" encoding of a graph as its adjacency matrix.

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left[\begin{array}{ccccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]
\end{array}
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Thus

We can then say the following

If the number of vertices is $m = \Omega\left(\sqrt{n}\right)$

We have then

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\sqrt{n}\left\{
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Then

The encoding size is

$$\sqrt{n} \times \sqrt{n} = n = |\langle G \rangle|$$

Then, I decide to go NAIVE!!!

The algorithm does the following

It lists the all permutations of the vertices of G and then checks each permutation to see if it is a Hamiltonian path.

Complexity

Performance analysis on the previous algorithm

- We have then $m = \Omega(\sqrt{n})$
- Then, for our naive algorithm produce $m!$ permutations.
- Then $\Omega(m!) = \Omega(\sqrt{n}!) = \Omega(2^{\sqrt{n}})$ **EXPONENTIAL TIME!!!**

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Observation of the class NP and co-NP

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- HAM-CYCLE is NP, thus NP class is not empty.
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The $co - NP$ class

The class called co-NP is the set of languages L such that $\bar{L} \in NP$

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We can restate the question of whether NP is closed under complement as whether $NP = co - NP$

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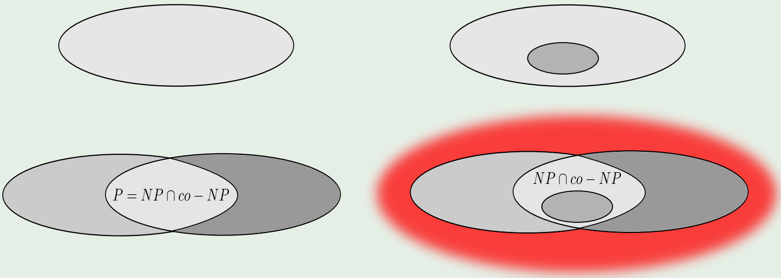
The four possibilities between the complexity classes

We have that



Exercises

From Cormen's book solve

- 34.2-1
- 34.2-2
- 34.2-5
- 34.2-6
- 34.2-9
- 34.2-10

Outline

Before entering reducibility

Why $P \neq NP$?

- Existence of NP-Complete problems.
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Rough definition

A problem M can be reduced to M' if any instance of M can be easily rephrased in terms of M'.

Formal definition

A language L is polynomial time reducible to a language L' written $L \leq_p L'$ if there exist a polynomial time computable function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ such that for all $x \in \{0,1\}^*$

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Graphically

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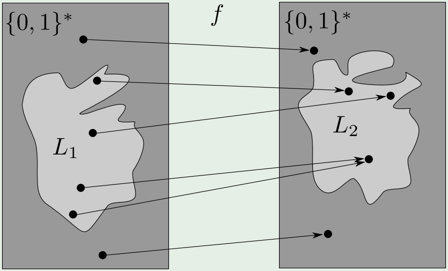

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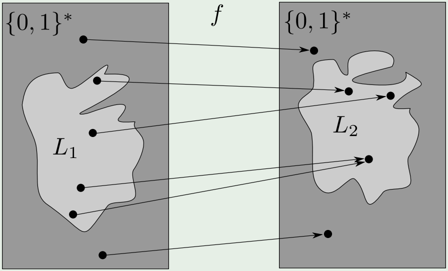

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Properties of f

Polynomial time reductions

Polynomial time reductions give us a powerful tool for proving that various languages belong to P.

How?

Lemma: If $L_1, L_2 \subseteq \{0,1\}^*$ are languages such that $L_1 \leq_p L_2$, then $L_2 \in P$ implies that $L_1 \in P$.

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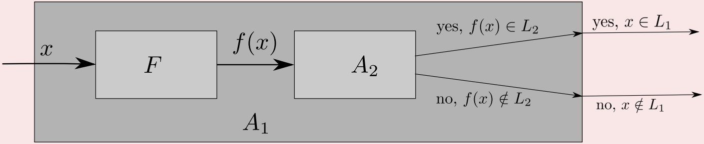

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A language $L \subseteq \{0,1\}^*$ is a NP-Complete problem (NPC) if:

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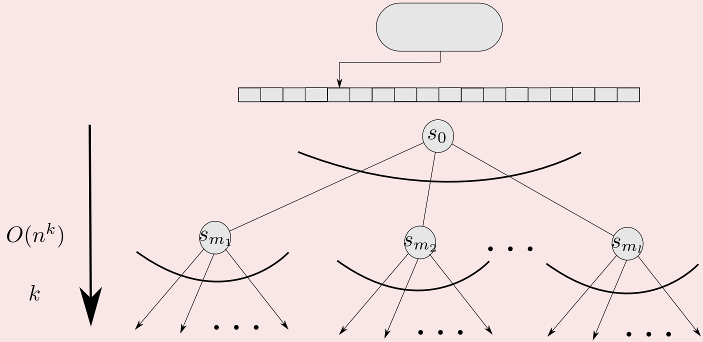

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Theorem

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Suppose that $L \in P$ and also that $L \in NPC$. For any $L' \in NP$, we have $L' \leq_p L$ by property 2 of the definition of NPC. Thus, by the previous Lemma, we have that $L' \in P$.

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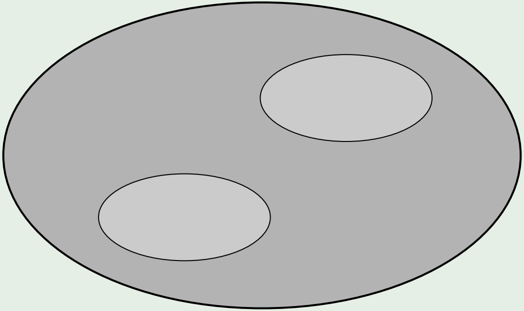
However

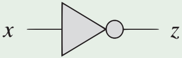
Most Theoretical Computer Scientist have the following view

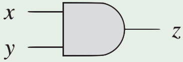


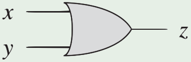
Outline

Our first NPC - Circuit Satisfiability

We have basic boolean combinatorial elements.







x	$\neg x$
0	1
1	0

x	y	$x \wedge y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x \vee y$
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Basic definition

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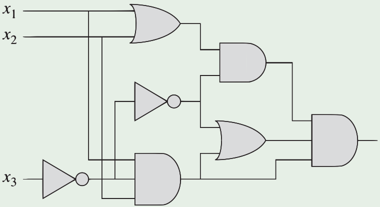
A boolean combinatorial circuit consist of one or more boolean combinatorial elements interconnected with wires.



Circuit satisfiability problem

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Given a boolean combinatorial circuit composed of AND, OR, and NOT gates, **Is it satisfiable? Output is ONE!!!**

Formally

$$CIRCUIT - SAT = \{\langle C\rangle \,|\, C \text{ is a satisfiable boolean combinatorial circuit}\}$$

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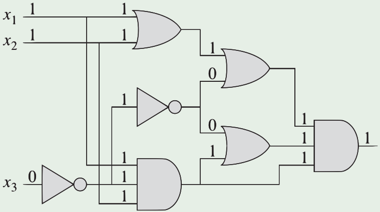

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We can represent the computation of A as a sequence of configurations.

- Start with configuration c_0, then finish with configuration $c_{T(n)}$.

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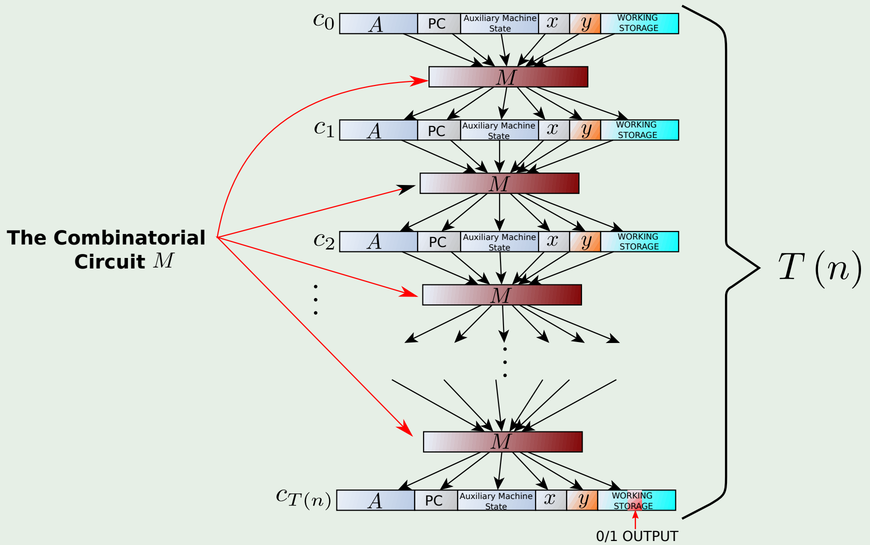
The idea



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- First, F modifies circuit C' in the following way:
  - It hardwires the inputs to C' corresponding to the program for A:
    - The initial program counter
    - The input x
    - The initial state of memory

Further

Something Notable

The only remaining inputs to the circuit correspond to the certificate y.

Then

- All outputs to the circuit are ignored, except the one bit of $c_{T(n)}$ corresponding to a computation on $A(x, y)$.

Because the only free input is the certificate y

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What we need to prove

First

- F correctly computes a reduction function f.
  - **C is satisfiable if and only if there is a certificate y such that $A(x, y) = 1$.**

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- We need to show that F runs in polynomial time.

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First, F correctly computes a reduction function f

We do the following

To show that F correctly computes a reduction function, let us suppose that there exists a certificate y of length $O(n^k)$ such that $A(x, y) = 1$.

$\Longleftarrow$

- If we apply the bits of y to the inputs of C, the output of C is $C(y) = A(x, y) = 1$. Thus, if a certificate exists, then C is satisfiable.

$\Longrightarrow$

- Now, suppose that C is satisfiable. Hence, there exists an input y to C such that $C(y) = 1$, from which we conclude that $A(x, y) = 1$.

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Next, we need to show that F runs in polynomial time

With respect to the polynomial reduction

The length of the input x is n, and the certificate y is $O(n^k)$.

Next

Circuit M implementing the computer hardware has polynomial size.

Properties

The circuit C consists of at most $t = O\left(n^k\right)$ copies of M.

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In conclusion

The language CIRCUIT-SAT is therefore at least as hard as any language in NP, and since it belongs to NP, it is NP-complete.

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Proving NP-Complete

Several theorems exist to make our life easier

We have the following

Lemma

If L is a Language such that $L' \leq_P L$ for some $L' \in NPC$, Then L is NP-Hard. Moreover, if $L \in NP$, then $L \in NPC$.

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So, we have the following strategy

Proceed as follows for proving that something is NP-Complete

1. Prove $L \in NP$.
2. Select a known NP-Complete language L'.
3. Describe an algorithm that computes function f mapping every instance $x \in \{0,1\}^*$ of L' to an instance $f(x)$ of L.
4. Prove that the function f satisfies: $x \in L'$ if and only if $f(x) \in L$ L for all $x \in \{0,1\}^*$.
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Exercises

From Cormen's book solve

- 34.3-1
- 34.3-2
- 34.3-5
- 34.3-6
- 34.3-7
- 34.3-8

Outline

Formula Satisfiability (SAT)

An instance of SAT is a boolean formula ϕ composed of

1. n boolean variables $x_1, x_2, ..., x_n$.
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Showing that SAT belongs to NP

Certificate

It consists of a satisfying assignment for an input formula ϕ.

Then A does the following

The verifying algorithm simply replaces each variable in the formula with its responding value and then evaluates the expression.

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Now, we try the mapping from CIRCUIT-SAT to SAT

Naïve algorithm

- We can use induction to express any boolean combinational circuit as a boolean formula.

Then

- We simply look at the gate that produces the circuit output and inductively express each of the gate's inputs as formulas.

Naively

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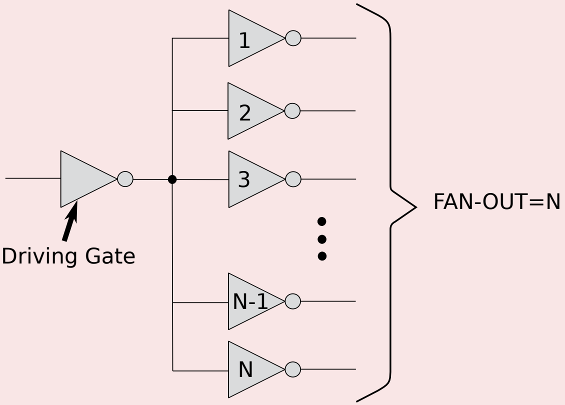
Problem

PROBLEM

What happens if the circuit fan out? I.e. shared sub-formulas can make the expression to grow exponentially!!!



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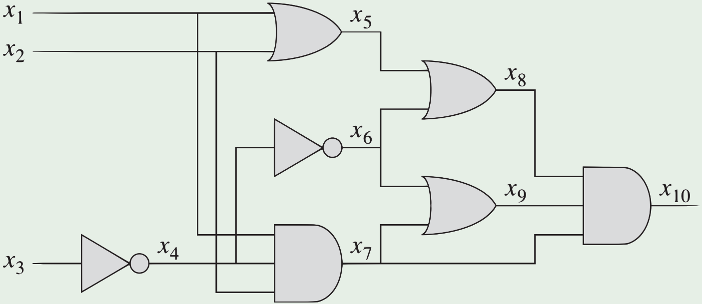
Use CIRCUIT-SAT

We a circuit C $\in$CIRCUIT-SAT



Thus, we have the following clauses

The new boolean formula

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\begin{aligned}
\phi = x_{10} &\wedge (x_4 \leftrightarrow \neg x_3)\\
&\wedge (x_5 \leftrightarrow (x_1 \vee x_2))\\
&\wedge (x_6 \leftrightarrow \neg x_4)\\
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Furthermore

Something Notable

Given that the circuit C is polynomial in size:

- it is straightforward to produce such a formula ϕ in polynomial time.

What do we want to prove?

Then

We want the following

$\Longrightarrow$

If C has a satisfying assignment then ϕ is satisfiable.

$\Longleftarrow$

If some assignment causes ϕ to evaluate to 1 then C is satisfiable.

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If C has a satisfying assignment, then each wire of the circuit has a well-defined value, and the output of the circuit is 1.

Meaning

Therefore, when we assign wire values to variables in ϕ, each clause of ϕ evaluates to 1, and thus the conjunction of all evaluates to 1.

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Conversely, if some assignment causes ϕ to evaluate to 1, the circuit C is satisfiable by an analogous argument.

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We have proved that

$CIRCUIT - SAT \leq_p SAT$

Outline

However!

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Problem: SAT is still too complex.

Solution: Use 3-CNF

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Definition

First

- A literal in a boolean formula is an occurrence of a variable or its negation.

Second

- A boolean formula is in conjunctive normal form, or CNF, if it is expressed as an AND of clauses, each of which is the OR of one or more literals.

Third

- A boolean formula is in 3-Conjunctive normal form, or 3-CNF, if each clause has exactly three distinct literals.

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Proof NP-Complete of 3-CNF

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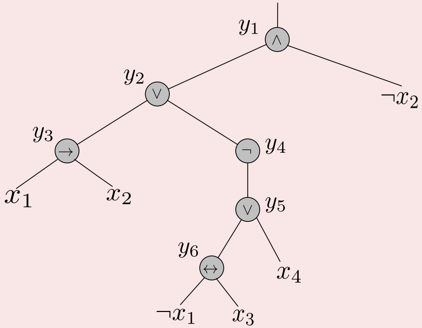

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This can be done by naming the nodes in the tree

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We still not have the disjunctive parts... What can we do?

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&\wedge (y_6 \leftrightarrow (\neg x_1 \leftrightarrow x_3))
\end{aligned}
$$

Problem

We still not have the disjunctive parts... What can we do?

Parsing allows us to parse the formula into ϕ'

This can be done by naming the nodes in the tree

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\begin{aligned}
\phi' = y_1 & \wedge (y_1 \leftrightarrow (y_2 \wedge \neg x_2)) \\
& \wedge (y_2 \leftrightarrow (y_3 \vee y_4)) \\
& \wedge (y_3 \leftrightarrow (x_1 \rightarrow x_2)) \\
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Proof

We can do the following

We can build the truth table of each clause ϕ'_i!

For example, the truth table of $\phi'_1 = y_1 \leftrightarrow (y_2 \wedge \neg x_2)$

y_1	y_2	x_3	$y_1 \leftrightarrow (y_2 \wedge \neg x_2)$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
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1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	1

From this, we have

Disjunctive normal form (or DNF)

In each of the zeros we put a conjunction that evaluate to ONE

y_1	y_2	x_3	$y_1 \leftrightarrow (y_2 \land \neg x_2)$	
1	1	1	0	$y_1 \land y_2 \land x_3$
1	1	0	1	$\cdots$
1	0	1	0	$y_1 \land \neg y_2 \land x_3$
1	0	0	0	$y_1 \land \neg y_2 \land \neg x_3$
0	1	1	1	$\cdots$
0	1	0	0	$\neg y_1 \land y_2 \land \neg x_3$
0	0	1	1	$\cdots$
0	0	0	1	$\cdots$

Then, we use disjunctions to put all them together

We have then an OR of AND's

$$I=(y_1\wedge y_2\wedge x_3)\vee(y_1\wedge\neg y_2\wedge x_3)\vee(y_1\wedge\neg y_2\wedge\neg x_3)\vee(\neg y_1\wedge y_2\wedge\neg x_3)$$

Thus, we have that $\neg I\equiv\phi_1'$

y_1	y_2	x_3	$y_1\leftrightarrow(y_2\wedge\neg x_2)$	I	$\neg I$
1	1	1	0	1	0
1	1	0	1	0	1
1	0	1	0	1	0
1	0	0	0	1	0
0	1	1	1	0	1
0	1	0	0	1	0
0	0	1	1	0	1
0	0	0	1	0	1

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We have then an OR of AND's

$$I=(y_1\land y_2\land x_3)\lor(y_1\land\neg y_2\land x_3)\lor(y_1\land\neg y_2\land\neg x_3)\lor(\neg y_1\land y_2\land\neg x_3)$$

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1	1	0	1	0	1
1	0	1	0	1	0
1	0	0	0	1	0
0	1	1	1	0	1
0	1	0	0	1	0
0	0	1	1	0	1
0	0	0	1	0	1

Using DeMorgan's laws

We obtain

$$\phi_1'' = (\neg y_1 \vee \neg y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee x_2)$$

Now, we need to include more literals as necessary

Given C_i as a disjunctive part of the previous formula.

- If C_i has 3 distinct literals, then simply include C_i as a clause of ϕ.

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If C_i has 2 distinct literals

if $C_i = (I_i \vee I_2)$, where I_1 and I_2 are literals, then include

$$(I_1 \vee I_2 \vee p) \wedge (I_1 \vee I_2 \vee \neg p)$$

as clauses of ϕ.

- Why?

$$(I_1 \vee I_2 \vee p) \wedge (I_1 \vee I_2 \vee \neg p) = (I_1 \vee I_2) \vee (p \wedge \neg p) =$$
$$(I_1 \vee I_2) \vee (F) = I_1 \vee I_2$$

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If C_i has just 1 distinct literal I

- Then include $(I \lor p \lor q) \land (I \lor p \lor \neg q) \land (I \lor \neg p \lor q) \land (I \lor \neg p \lor \neg q)$ as clauses of ϕ.
- Why?

$$
\begin{aligned}
(I \lor p \lor q) \land (I \lor p \lor \neg q) \land \dots \\
(I \lor \neg p \lor q) \land (I \lor \neg p \lor \neg q) &= I \lor [(p \lor q) \land (p \lor \neg q) \land \dots \\
&\quad (\neg p \lor q) \land (\neg p \lor \neg q)] \\
&= I \lor [p \lor (q \land \neg q) \land \dots \\
&\quad (\neg p \lor (q \land \neg q))] \\
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$$

Finally, we need to prove the polynomial reduction

First

- Constructing ϕ' from ϕ introduces at most 1 variable and 1 clause per connective in ϕ.

Second

- Constructing ϕ'' from ϕ' can introduce at most 8 clauses into ϕ'' for each clause from ϕ', since each clause of ϕ'' has at most 3 variables, and the truth table for each clause has at most $2^3 = 8$ rows.

Third

- The construction of ϕ''' from ϕ' introduces at most 4 clauses into ϕ''' for each clause of ϕ''.

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Thus

$$SAT \leq_p 3-CNF$$

Theorem 34.10

Satisfiability of boolean formulas in 3-conjunctive normal form is NP-complete.

Now the next problem

The Clique Problem.

Finally

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Excercises

From Cormen's book solve

- 34.4-1
- 34.4-2
- 34.4-5
- 34.4-6
- 34.4-7

Outline

The Clique Problem

Definition

A clique in an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices, each pair of which is connected by an edge in E.

As a decision problem

$CLIQUE = \{< G, k > \mid G \text{is a graph with a clique of size } k\}$

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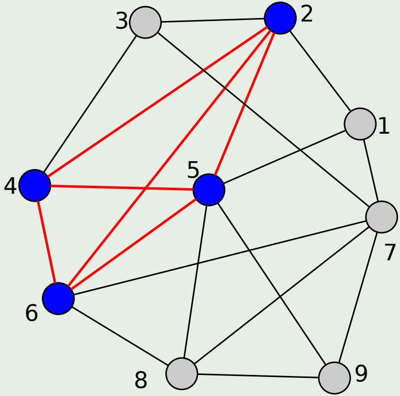
As a decision problem

$CLIQUE = \{< G, k > \mid G$is a graph with a clique of size $k\}$

Example

A Clique of size $k = 4$



The clique problem is NP-Complete

Theorem 34.11

The clique problem is NP-Complete.

Proof

1. To show that $CLIQUE \in NP$, for a given graph $G = (V, E)$ we use the set $V' \in V$ of vertices in the clique as certificate for G.

Thus

This is can be done in polynomial time because we only need to check all possibles pairs of $u, v \in V'$, which takes $|V'|(|V'| - 1)$.

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Proof

Now, we only need to prove that the problem is NP-Hard

Which is surprising, after all we are going from logic to graph problems!!!

Now

We start with an instance of 3-CNF-SAT

- $C_1 \wedge C_2 \wedge \ldots \wedge C_k$ a boolean 3-CNF formula with k clauses.

We know for each $1 \leq r \leq k$

$$C_r = l_1^r \vee l_2^r \vee l_3^r$$

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Then

Now, we construct the following graph $G = (V, E)$

We place a triple of vertices v_1^r, v_2^r, v_3^r for each $C_r = l_1^r \vee l_2^r \vee l_3^r$.

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Now, we construct the following graph $G = (V, E)$

We place a triple of vertices v_1^r, v_2^r, v_3^r for each $C_r = l_1^r \lor l_2^r \lor l_3^r$.

We put an edge between two vertices v_i^r and v_j^s, if

- v_i^r and v_j^s are in different triples i.e. $r \neq s$.
- Their corresponding literals are consistent i.e. l_i^r is not the negation of l_j^s

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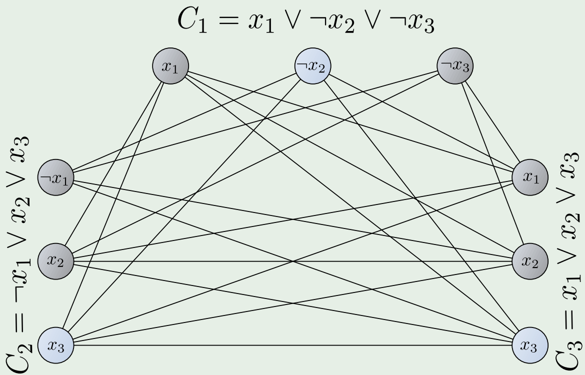
Example

For $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$



Now, we show that the transformation ϕ into G is a reduction

We start with the $\Longrightarrow$

- Suppose ϕ has a satisfying assignment.
- Thus each clause C_r contains at least one literal l_i^r mapping to 1, which corresponds to a vertex v_i^r.

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Now, we pick each of those literals

We finish with a set V' of such literals.

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V' is a clique, how?

Given two vertices v_i^r and $v_j^s \in V'$, with $r \neq s$, such that the corresponding literals l_i^r and l_j^s map to 1 by the satisfying assignment.

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Then

This two literals

They cannot be complements.

Finally

By construction of G, the edge $\left(v_i^r, v_j^s\right)$ belongs to E.

Thus

We have a clique of size k in the graph $G = (V, E)$.

Then

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We now prove $\Longleftarrow$

Conversely

Suppose that G has a clique V' of size k.

Did you notice?

No edges in G connect vertices in the same triple, thus V' contains one vertex per triple.

Now

Now, we assign 1 to each literal l_i^r such that $v_i^r \in V'$

- Notice that we cannot assign 1 to both a literal and its complement by construction.

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We have with that assignment

That each clause C_r is satisfied, thus ϕ is satisfied!!!

Note

Any variables that do not correspond to a vertex in the clique may be set arbitrarily.

Then

$$3 - CNF \leq_p CLIQUE$$

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Remarks

Something Notable

We have reduced an arbitrary instance of 3-CNF-SAT to an instance of CLIQUE with a particular structure.

Thus

It is possible to think that we have shown only that CLIQUE is NP-hard in graphs in which the vertices are restricted to occur in triples and in which there are no edges between vertices in the same triple.

Actually this is true

- But it is enough to prove that CLIQUE is NP-hard.
- Why? If we had a polynomial-time algorithm that solved CLIQUE in the general sense, we will solve in polynomial time the restricted version.

Remarks

Something Notable

We have reduced an arbitrary instance of 3-CNF-SAT to an instance of CLIQUE with a particular structure.

Thus

It is possible to think that we have shown only that CLIQUE is NP-hard in graphs in which the vertices are restricted to occur in triples and in which there are no edges between vertices in the same triple.

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In the opposite approach

Reducing instances of 3-CNF-SAT with a special structure to general instances of CLIQUE would not have sufficed.

Why not?

- Perhaps the instances of 3-CNF-SAT that we chose to reduce from were "easy," not reducing an NP-Hard problem to CLIQUE.
- Observe also that the reduction used the instance of 3-CNF-SAT, but not the solution.

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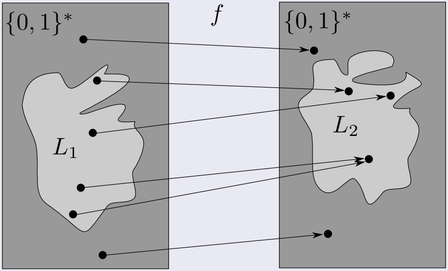
Thus

This would have been a serious error

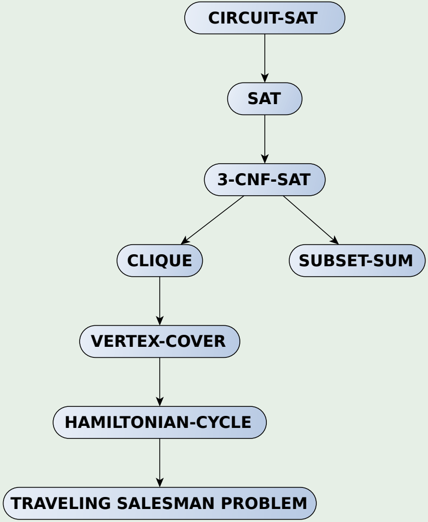
Remember the mapping:



Outline

Now, we have

Family of NP-Complete Problems



Excercises

From Cormen's book solve

- 34.5-1
- 34.5-2
- 34.5-3
- 34.5-4
- 34.5-5
- 34.5-7
- 34.5-8