Analysis of Algorithms

Multi-threaded Algorithms

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Outline

Cinvestav

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Multi-Threaded Algorithms

Motivation

- Until now, our serial algorithms are quite suitable for running on a single processor system.
- However, multiprocessor algorithms are ubiquitous:
    - Therefore, extending our serial models to a parallel computation model is a must.

    - Shared Memory
    - Message Passing
    - Etc.

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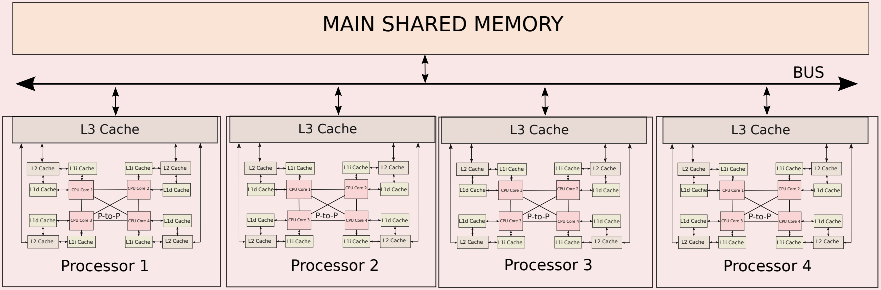
The Model to Be Used

Symmetric Multiprocessor

The model that we will use is the Symmetric Multiprocessor (SMP) where a shared memory exists.



Dynamic Multi-Threading

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- In reality it can be difficult to handle multi-threaded programs in a SMP.
- Thus, we will assume a simple concurrency platform that handles all the resources:
    - Schedules
    - Memory
    - Etc
- It is Called **Dynamic Multi-threading**.

- Sync
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When called before a procedure, the parent procedure may continue to execute in parallel.

- The Scheduler decide which computations should run concurrently

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- The keyword **spawn** does not say anything about concurrent execution, but it can happen.
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SYNC AND PARALLEL

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The keyword sync indicates that the procedure must wait for all its spawned children to complete.

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This operation applies to loops, which make possible to execute the body of the loop in parallel.

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1. Introduction
    - Why Multi-Threaded Algorithms?
2. Model To Be Used
    - Symmetric Multiprocessor
    - Operations
    - Example
3. Computation DAG
    - Introduction
4. Performance Measures
    - Introduction
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    - Work and Span Laws
    - Speedup and Parallelism
    - Greedy Scheduler
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    - Parallel Fibonacci
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7. Exercises
    - Some Exercises you can try!!!

A Classic Parallel Piece of Code: Fibonacci Numbers

Fibonacci's Definition

- $F_0 = 0$
- $F_1 = 1$
- $F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

Naive Algorithm

```
$Fibonacci(n)$
if $n \leq 1$ then
        return $n$
else $x = Fibonacci(n-1)$
        $y = Fibonacci(n-2)$
        return $x + y$
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Time Complexity

Recursion and Complexity

- Recursion $T(n) = T(n-1) + T(n-2) + \Theta(1)$.
- Complexity $T(n) = \Theta(F_n) = \Theta(\phi^n)$, $\phi = \frac{1+\sqrt{5}}{2}$.

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- Complexity $T(n)=\Theta\left(F_n\right)=\Theta\left(\phi^n\right)$, $\phi=\frac{1+\sqrt{5}}{2}$.

There is a Better Way

We can order the first tree numbers in the sequence as

$$\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$\begin{aligned} \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} F_3 & F_2 \\ F_2 & F_1 \end{pmatrix} \end{aligned}$$

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How do we compute a complexity? Computation DAG

Definition

A directed acyclic $G = (V, E)$ graph where

- The vertices V are sets of instructions.
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- Thus, V represents a set of strands and E represents dependencies between strands induced by parallel control.
- A strand of maximal length will be called a **thread**.

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Example: PFibonacci(4)

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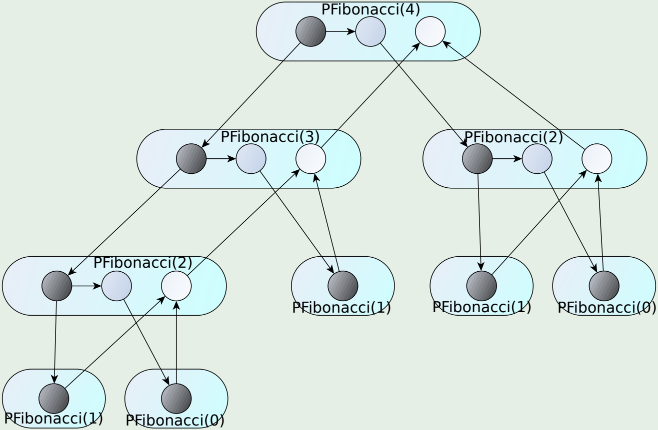

Edge Classification

Continuation Edge

A **continuation edge** (u, v) connects a thread u to its successor v within the same procedure instance.

Spawned Edge

When a thread u spawns a new thread v, then (u, v) is called a **spawned edge.**

Call Edges

Call edges represent normal procedure call.

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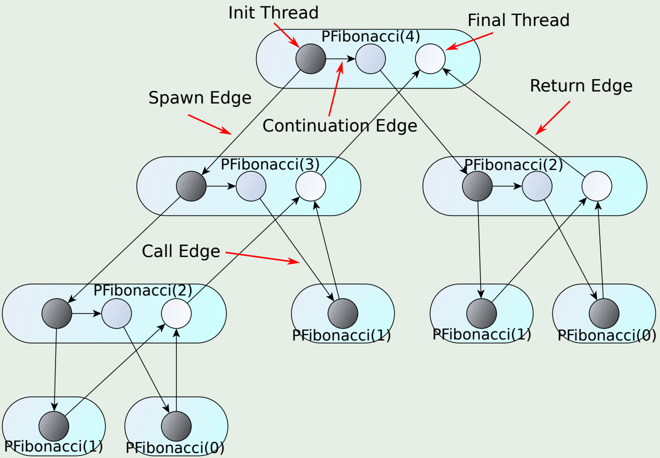

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Performance Measures

WORK

The work of a multi-threaded computation is the total time to execute the entire computation on **one processor.**

$$Work = \sum_{i \in I} Time\left(Thread_i\right)$$

SPAN

The span is the longest time to execute the strands along any path of the DAG.

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- In a DAG which each strand takes unit time, the span equals the number of vertices on a longest or **critical path** in the DAG.

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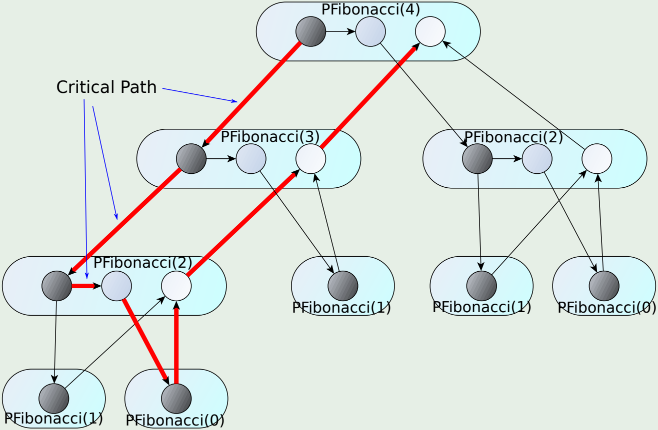

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- In Fibonacci(4), we have
    - 17 threads.
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We have that

- Assuming unit time
    - WORK=17 time units
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Running Time Classification

Single Processor

- T_1 running time on a single processor.

Multiple Processors

- T_p running time on P processors.

Unlimited Processors

- T_∞ running time on unlimited processors, also called the span, if we run each strand on its own processor.

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Work Law

Definition

- In one step, an ideal parallel computer with P processors can do:
  - At most P units of work.
  - Thus in T_P time, it can perform at most PT_P work.

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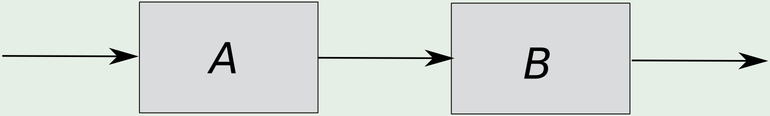
Serial Computations



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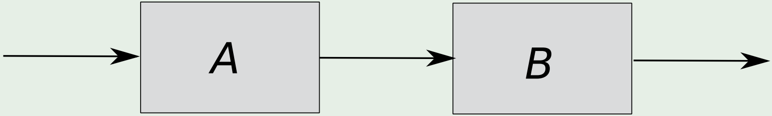

Note

- Work: $T_1 \left(A \cup B \right) = T_1 \left(A \right) + T_1 \left(B \right)$.
- Span: $T_\infty \left(A \cup B \right) = T_\infty \left(A \right) + T_\infty \left(B \right)$.

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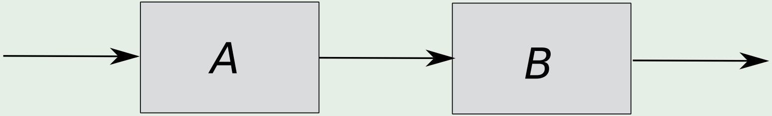

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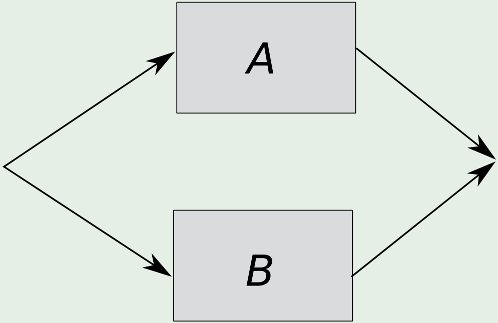

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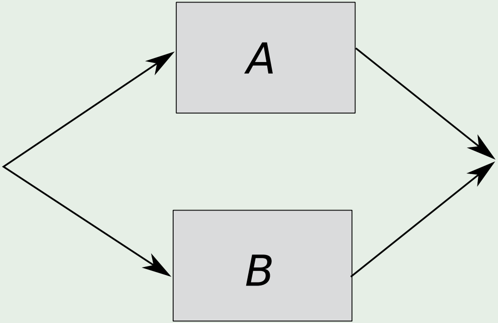

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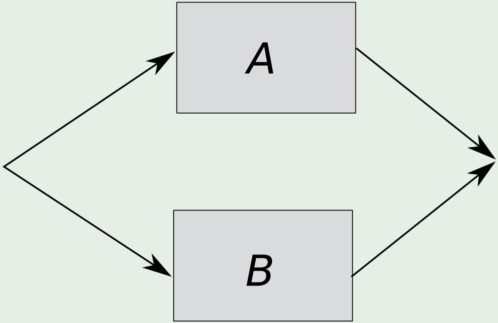
Work Calculations: Parallel

Parallel Computations



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Speedup and Parallelism

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- Then, by work law $\frac{T_1}{T_P} \leq P$. Thus, the speedup on P processors can be at most P.

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Greedy Scheduler Theorem and Corollaries

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On an ideal parallel computer with P processors, a greedy scheduler executes a multi-threaded computation with work T_1 and span T_∞ in time $T_P \leq \frac{T_1}{P} + T_\infty$.

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The running time T_P of any multi-threaded computation scheduled by a greedy scheduler on an ideal parallel computer with P processors is within a factor of 2 of optimal.

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Let T_P be the running time of a multi-threaded computation produced by a greedy scheduler on an ideal parallel computer with P processors, and let T_1 and T_∞ be the work and span of the computation, respectively. Then, if $P \ll \frac{T_1}{T_\infty}$ (Much Less), we have $T_P \approx \frac{T_1}{P}$, or equivalently, a speedup of approximately P .

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Race Conditions

Determinacy Race

A determinacy race occurs when two logically parallel instructions access the same memory location and at least one of the instructions performs a write.

Example

Race-Example()

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$x = 0$
parallel for $i = 1$ to 3 do
    $x = x + 1$
print $x$
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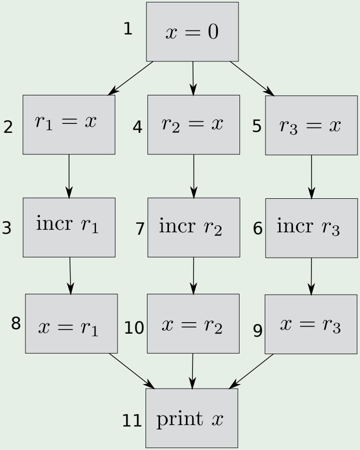

step	x	r_1	r_2	r_3
1	0			
2	0	0		
3	0	1		
4	0	1	0	
5	0	1	0	0
6	0	1	0	1
7	0	1	1	1
8	1	1	1	1
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10	1	1	1	1

Cinvestav

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Although, this is of great importance is beyond the scope of this class:

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Example of Complexity: PFibonacci

Complexity

$$T_\infty(n) = \max\left\{T_\infty(n-1), T_\infty(n-2)\right\} + \Theta(1)$$

Finally

$$T_\infty(n) = T_\infty(n-1) + \Theta(1) = \Theta(n)$$

Parallelism

$$\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{\phi^n}{n}\right)$$

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Outline

1. Introduction
    - Why Multi-Threaded Algorithms?
2. Model To Be Used
    - Symmetric Multiprocessor
    - Operations
    - Example
3. Computation DAG
    - Introduction
4. Performance Measures
    - Introduction
    - Running Time Classification
5. Parallel Laws
    - Work and Span Laws
    - Speedup and Parallelism
    - Greedy Scheduler
    - Scheduling Rises the Following Issue
6. **Examples**
    - Parallel Fibonacci
    - **Matrix Multiplication**
    - Parallel Merge-Sort
7. Exercises
    - Some Exercises you can try!!!

Matrix Multiplication

Trick

To multiply two $n \times n$ matrices, we perform 8 matrix multiplications of $\frac{n}{2} \times \frac{n}{2}$ matrices and one addition $n \times n$ of matrices.

Idea

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

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Any Idea to Parallelize the Code?

What do you think?

Did you notice the multiplications of sub-matrices?

Then What?

We have for example $A_{11}B_{11}$ and $A_{12}B_{21}$!!!

We can do the following

$$A_{11}B_{11} + A_{12}B_{21}$$

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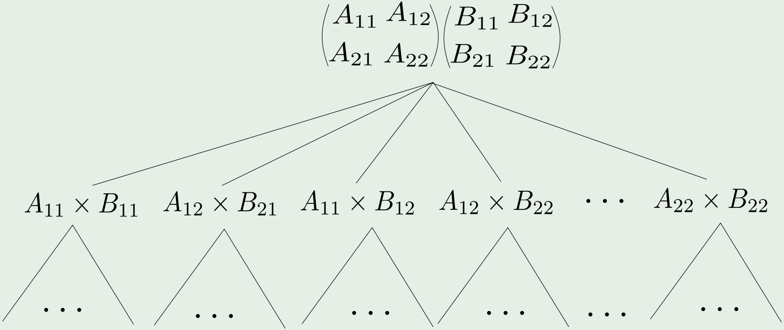
The use of the recursion!!!

As always our friend!!!



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Pseudo-code of Matrix-Multiply

$Matrix-Multiply(C, A, B, n)$ **// The result of $A \times B$ in C with n a power of 2 for simplicity**

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Calculating the products in

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Using Recursion and Parallel Computations

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A barrier to wait until all the parallel computations are done!!!

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        spawn $Matrix - Add(C_{12}, T_{12}, n/2)$
        spawn $Matrix - Add(C_{21}, T_{21}, n/2)$
        $Matrix - Add(C_{22}, T_{22}, n/2)$
        sync
```

Matrix ADD

Matrix Add Code

```
$Matrix - Add(C, T, n)$
// Add matrices $C$ and $T$ in-place to produce $C = C + T$
if $(n == 1)$
        $C[1,1] = C[1,1] + T[1,1]$
else
        Partition $C$ and $T$ into $\frac{n}{2} \times \frac{n}{2}$ sub-matrices
        spawn $Matrix - Add\left(C_{11}, T_{11}, {}^{n}/2\right)$
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```

Explanation

Line 1 - 2

Stops the recursion once you have only two numbers to multiply

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To Partition

- $C = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{pmatrix}$
- $T = \begin{pmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$

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In lines 5 to 8

We do the following sum in parallel!!!

$$\underbrace{\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{pmatrix}}_{C} + \underbrace{\begin{pmatrix} A_{12}B_{21} & A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}}_{T}$$

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Calculating Complexity of Matrix Multiplication

Work of Matrix Multiplication

The work of $T_1(n)$ of matrix multiplication satisfies the recurrence:

$$T_1(n) = \underbrace{8\,T_1\left(\frac{n}{2}\right)}_{\text{The sequential product}} + \underbrace{\Theta\left(n^2\right)}_{\text{The sequential sum}} = \Theta\left(n^3\right).$$

Calculating Complexity of Matrix Multiplication

Span of Matrix Multiplication

$$T_{\infty}(n) = \underbrace{T_{\infty}\left(\frac{n}{2}\right)}_{\text{The parallel product}} + \underbrace{\Theta(\log n)}_{\text{The parallel sum}} = \Theta\left(\log^2 n\right)$$

This is because:

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This is because:

- $T_{\infty}\left(\frac{n}{2}\right)$ Matrix Multiplication is taking $\frac{n}{2} \times \frac{n}{2}$ matrices at the same time because parallelism.
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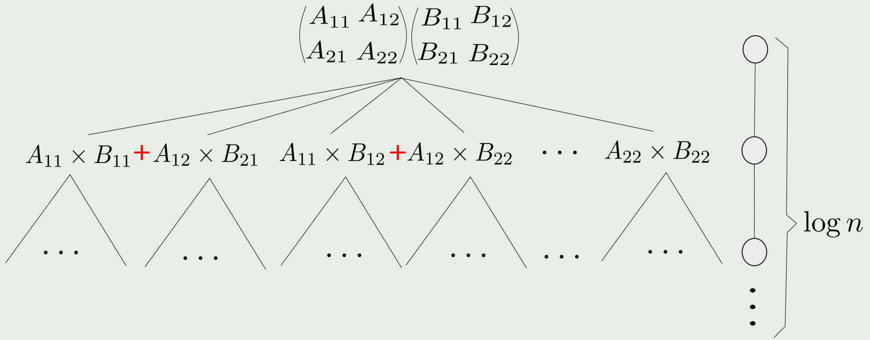
Collapsing the sum

Parallel Sum



How much Parallelism?

The Final Parallelism in this Algorithm is

$$\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n^3}{\log^2 n}\right)$$

Quite A Lot!!!

Outline

Merge-Sort : The Serial Version

We have

$Merge - Sort\,(A, p, r)$

Observation: **Sort elements in** $A\,[p...r]$

```
if $(p < r)$ then
    $q = \left\lfloor (p+r)/2 \right\rfloor$
    $Merge - Sort\,(A, p, q)$
    $Merge - Sort\,(A, q+1, r)$
    $Merge\,(A, p, q, r)$
```

Merge-Sort : The Parallel Version

We have

$Merge-Sort\left(A,p,r\right)$

Observation: **Sort elements in** $A\left[p...r\right]$

```
1. if $(p < r)$ then
2.         $q = \lfloor ^{(p+r)}/_{2} \rfloor$
3.         spawn $Merge-Sort\left(A,p,q\right)$
4.         $Merge-Sort\left(A,q+1,r\right)$ // Not necessary to spawn this
5.         sync
6.         $Merge\left(A,p,q,r\right)$
```

Calculating Complexity of This simple Parallel Merge-Sort

Work of Merge-Sort

- The work of $T_1(n)$ of this Parallel Merge-Sort satisfies the recurrence:

$$T_1(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T_1\left(\frac{n}{2}\right) + \Theta(n) & \text{otherwise} \end{cases} = \Theta(n \log n)$$

Because the Master Theorem Case 2.

Calculating Complexity of This simple Parallel Merge-Sort

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$$T_\infty(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ T_\infty\left(\frac{n}{2}\right) + \Theta(n) & \text{otherwise} \end{cases}$$

We have then

- $T_\infty\left(\frac{n}{2}\right)$ sort is taking two sorts at the same time because parallelism.
- **Then, $T_\infty(n) = \Theta(n)$ because the Master Theorem Case 3.**

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- The work of $T_1(n)$ of this Parallel Merge-Sort satisfies the recurrence:

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- $T_\infty\left(\frac{n}{2}\right)$ sort is taking two sorts at the same time because parallelism.
- **Then,** $T_\infty(n) = \Theta(n)$ **because the Master Theorem Case 3.**

How much Parallelism?

The Final Parallelism in this Algorithm is

$$\frac{T_1(n)}{T_\infty(n)} = \Theta(\log n)$$

NOT NOT A Lot!!!

Can we improve this?

We have a problem

We have a bottleneck!!! Where?

Yes in the Merge part!!!

We need to improve that bottleneck!!!

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We have a problem

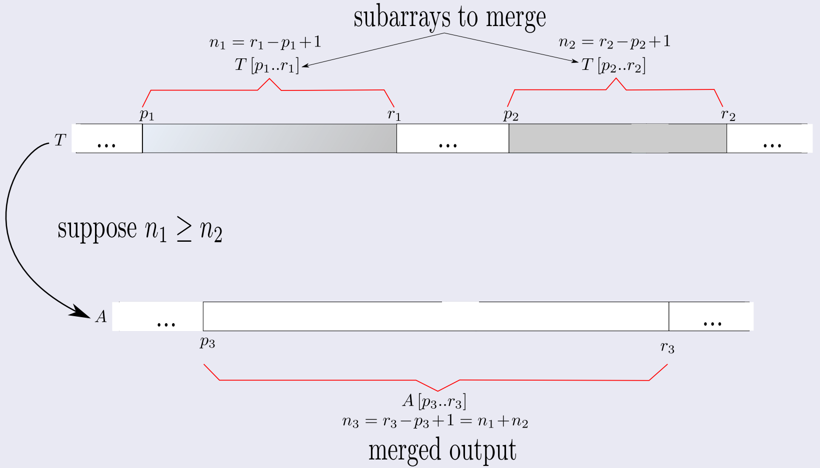
We have a bottleneck!!! Where?

Yes in the Merge part!!!

We need to improve that bottleneck!!!

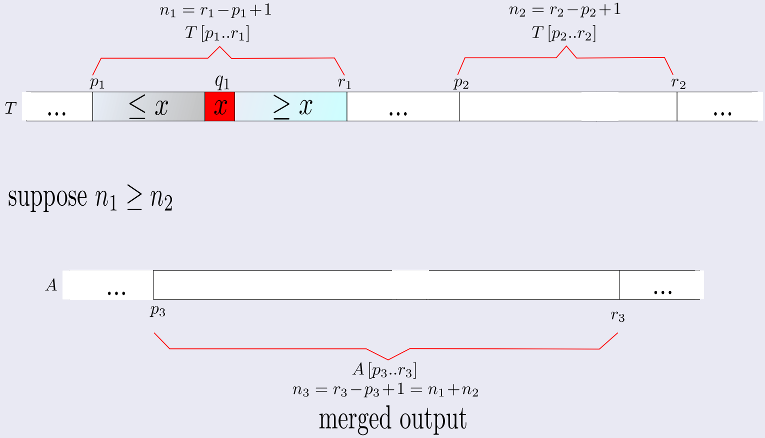
Parallel Merge

Example: Here, we use and intermediate array T



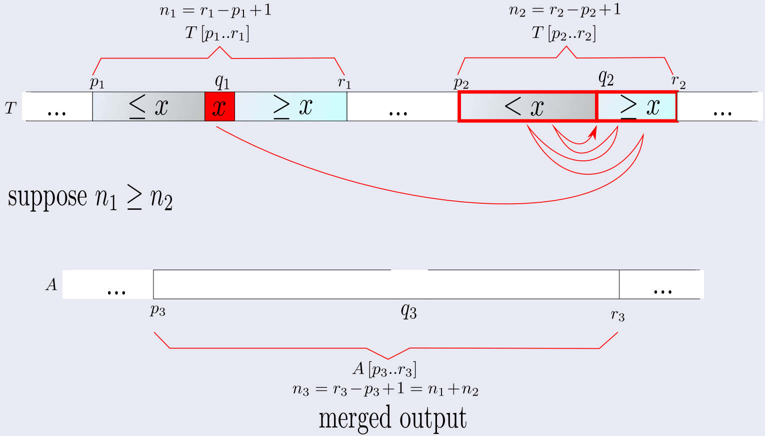
Parallel Merge

Step 1. Find $x = T[q_1]$ where $q_1 = \lfloor (p_1+r_1)/2 \rfloor$ or the midpoint in $T[p_1..r_1]$



Parallel Merge

Step 2. Use Binary Search in $T[p_1..r_1]$ to find q_2



Then

So that if we insert x between $T[q_2 - 1]$ and $T[q_2]$

$T\left[\begin{array}{ccccccc} p_1 & \cdots & q_2-1 & x & q_2 & \cdots & r_1 \end{array}\right]$ is sorted

Binary Search

It takes a key x and a sub-array $T[p..r]$ and it does

1. If $T[p..r]$ is empty $r < p$, then it returns the index p.
2. if $x \leq T[p]$, then it returns p.
3. if $x > T[p]$, then it returns the largest index q in the range $p < q \leq r + 1$ such that $T[q-1] < x$.

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Binary Search Code

```
BINARY-SEARCH($x$, $T$, $p$, $r$)
$low = p$
$high = \max\{p, r+1\}$
while $low < high$
    $mid = \left\lfloor \frac{log+high}{2} \right\rfloor$
    if $x \leq T[mid]$
        $high = mid$
    else $low = mid + 1$
return $high$
```

Binary Search Code

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Binary Search Code

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    else low = mid + 1
return high
```

Binary Search Code

BINARY-SEARCH(x, T, p, r)

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$low = p$
$high = \max\{p, r+1\}$
while $low < high$
    $mid = \left\lfloor \frac{log + high}{2} \right\rfloor$
    if $x \leq T[mid]$
        $high = mid$
    else $low = mid + 1$
return $high$
```

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Binary Search Code

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$low = p$
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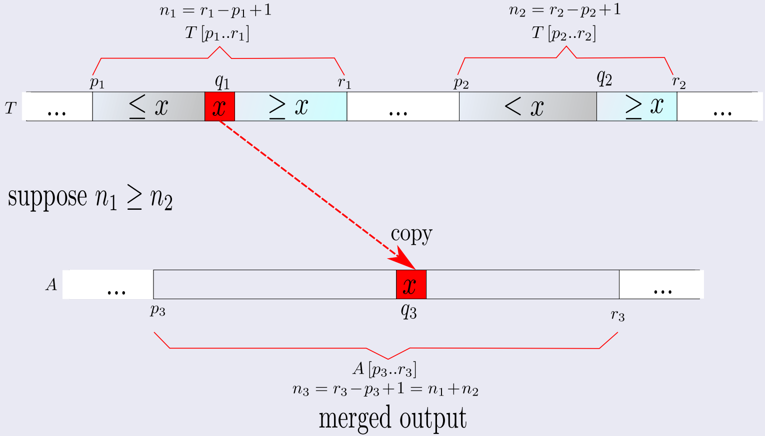
while $low < high$
        $mid = \left\lfloor \frac{log + high}{2} \right\rfloor$
        if $x \leq T[mid]$
                $high = mid$
        else $low = mid + 1$
return $high$
```

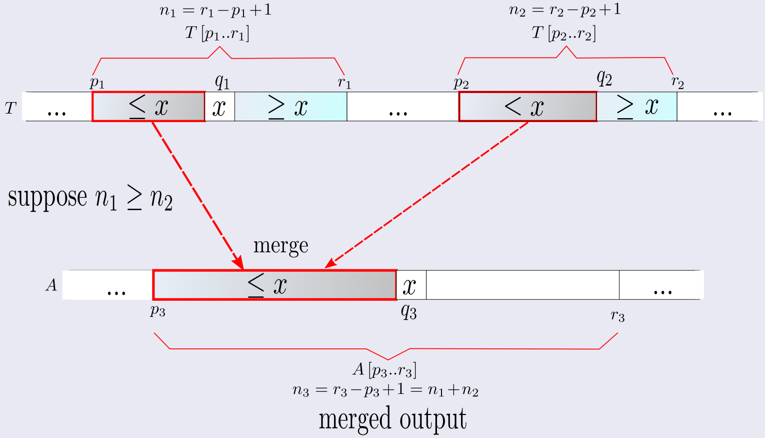
Parallel Merge

Step 3. Copy x in $A[q_3]$ where $q_3 = p_3 + (q_1 - p_1) + (q_2 - p_2)$



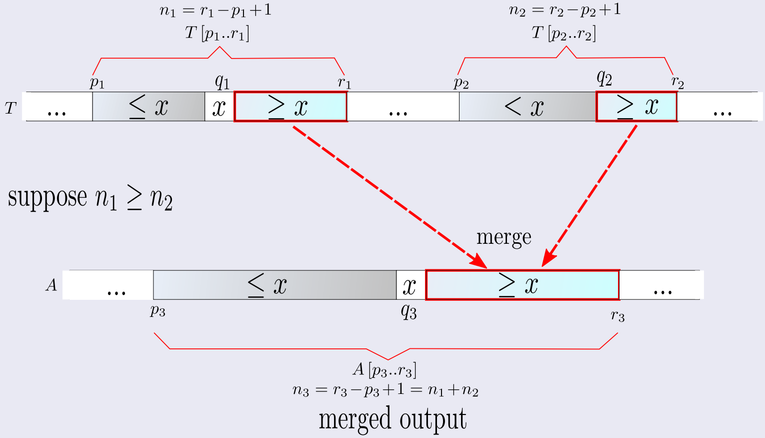
Parallel Merge

Step 4. Recursively merge $T[p_1..q_1-1]$ and $T[p_2..q_2-1]$ and place result into $A[p_3..q_3-1]$



Parallel Merge

Step 5. Recursively merge $T[q_1+1..r_1]$ and $T[q_2..r_2]$ and place result into $A[q_3+1..r_3]$



The Final Code for Parallel Merge

```
$Par - Merge\left(T, p_1, r_1, p_2, r_2, A, p_3\right)$
    $n_1 = r_1 - p_1 + 1$, $n_2 = r_2 - p_2 + 1$
    if $n_1 < n_2$
        Exchange $p_1 \leftrightarrow p_2, r_1 \leftrightarrow r_2, n_1 \leftrightarrow n_2$
    if $(n_1 == 0)$
        return
    else
        $q_1 = \lfloor (p_1+r_1)/2 \rfloor$
        $q_2 = BinarySearch\left(T\left[q_1\right], T, p_2, r_2\right)$
        $q_3 = p_3 + (q_1 - p_1) + (q_2 - p_2)$
        $A\left[q_3\right] = T\left[q_1\right]$
        spawn $Par - Merge\left(T, p_1, q_1 - 1, p_2, q_2 - 1, A, p_3\right)$
        $Par - Merge\left(T, q_1 + 1, r_1, q_2 + 1, r_2, A, q_3 + 1\right)$
        sync
```

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$Par - Merge\,(T, p_1, r_1, p_2, r_2, A, p_3)$
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    else
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        $q_2 = BinarySearch\,(T[q_1], T, p_2, r_2)$
        $q_3 = p_3 + (q_1 - p_1) + (q_2 - p_2)$
        $A[q_3] = T[q_1]$
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Explanation

Line 1

Obtain the length of the two arrays to be merged

Line 2: If one is larger than the other

We exchange the variables to work the largest element!!! In this case we make $n_1 \geq n_2$

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if $n_1 == 0$ return nothing to merge!!!

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Line 10

It copies $T[q_1]$ directly into $A[q_3]$

Line 11 and 12

They are used to recurse using nested parallelism to merge the sub-arrays less and greater than x.

Line 13

The sync is used to ensure that the subproblems have completed before the procedure returns.

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First the Span Complexity of **Parallel Merge**: $T_\infty(n)$

Suppositions

- $n = n_1 + n_2$

What case should we study?

Remember $T_\infty(n) = \max\{T_\infty(n_1) + T_\infty(n_2)\}$

We notice then that

Because lines 3-6 $n_2 \leq n_1$

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Span Complexity of the **Parallel Merge** with One Processor: $T_1(n)$

Then

$$2n_2 \leq n_1 + n_2 = n \Longrightarrow n_2 \leq {}^{n}/_{2}$$

Span Complexity of the **Parallel Merge** with One Processor: $T_1(n)$

Then

$$2n_2 \leq n_1 + n_2 = n \Longrightarrow n_2 \leq {}^{n}/_{2}$$

Thus

In the worst case, a recursive call in lines 11 merges:

- $\left\lfloor \frac{n_1}{2} \right\rfloor$ elements of $T\left[p_1 \ldots r_1\right]$ (Remember we are halving the array by mid-point).
- With all n_2 elements of $T\left[p_2 \ldots r_2\right]$.

Span Complexity of the **Parallel Merge** with One Processor: $T_1(n)$

Then

$$2n_2 \leq n_1 + n_2 = n \Longrightarrow n_2 \leq {}^{n}/_{2}$$

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Thus, the number of elements involved in such a call is

$$
\begin{aligned}
\left\lfloor \frac{n_1}{2} \right\rfloor + n_2 &\leq \frac{n_1}{2} + \frac{n_2}{2} + \frac{n_2}{2} \\
&\leq \frac{n_1}{2} + \frac{n_2}{2} + \frac{n/2}{2} \\
&= \frac{n_1 + n_2}{2} + \frac{n}{4} \\
&\leq \frac{n}{2} + \frac{n}{4} = \frac{3n}{4}
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Span Complexity of the **Parallel Merge** with One Processor: $T_1(n)$

Knowing that the Binary Search takes

$$\Theta(\log n)$$

We get the span for parallel merge

$$T_\infty(n) = T_\infty\left(\frac{3n}{4}\right) + \Theta(\log n)$$

This can can be solved using the exercise 4.6-2 in the Cormen's Book

$$T_\infty(n) = \Theta\left(\log^2 n\right)$$

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Calculating Work Complexity of **Parallel Merge**

Ok!!! We need to calculate the WORK

$$T_1(n) = \Theta(Something)$$

Thus

We need to calculate the upper and lower bound.

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Work of Parallel Merge

The work of $T_1(n)$ of this Parallel Merge satisfies:

$$T_1(n) = \Omega(n)$$

- Because each of the n elements must be copied from array T to array A.

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- $\frac{n}{4}$ elements when we have we have the worst case of $\left\lfloor \frac{n_1}{2} \right\rfloor + n_2$ in the other merge.
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Then

Then, for some $\alpha \in \left[\frac{1}{4}, \frac{3}{4}\right]$, then we have the following recursion for the Parallel Merge when we have one processor:

$$T_1(n) = \underbrace{T_1(\alpha n) + T_1((1-\alpha)n)}_{\text{Merge Part}} + \underbrace{\Theta(\log n)}_{\text{Binary Search}}$$

Remark: α varies at each level of the recursion!!!

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Calculating Work Complexity of **Parallel Merge**

Then

Assume that $T_1(n) \leq c_1 n - c_2 \log n$ for positive constants c_1 and c_2.

We have then using c_3 for $\Theta(\log n)$

$$T_1(n) \leq T_1(\alpha n) + T_1((1-\alpha)n) + c_3 \log n$$

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$$\begin{aligned} T_1(n) &\leq T_1(\alpha n) + T_1((1-\alpha) n) + c_3 \log n \\ &\leq c_1 \alpha n - c_2 \log(\alpha n) + c_1 (1-\alpha) n - c_2 \log((1-\alpha) n) + c_3 \log n \\ &= c_1 n - c_2 \log(\alpha(1-\alpha)) - 2 c_2 \log n + c_3 \log n \text{ (splitting elements)} \\ &= c_1 n - c_2 (\log n + \log(\alpha(1-\alpha))) - (c_2 - c_3) \log n \\ &\leq c_1 n - (c_2 - c_3) \log n \text{ because } \log n + \log(\alpha(1-\alpha)) > 0 \end{aligned}$$

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\end{aligned}$$

Calculating Work Complexity of **Parallel Merge**

Now, we have that given $0 < \alpha(1-\alpha) < 1$

We have $\log\left(\alpha(1-\alpha)\right) < 0$

Thus, making n large enough

$$\log n + \log\left(\alpha(1-\alpha)\right) > 0 \tag{1}$$

Then

$$T_1\left(n\right) \leq c_1 n - \left(c_2 - c_3\right) \log n$$

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Calculating Work Complexity of **Parallel Merge**

Now, we choose c_2 and c_3 such that

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We have that

$$T_1(n) \leq c_1 n = O(n)$$

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$$T_1(n) = \Theta(n)$$

The parallelism of Parallel Merge

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Then What is the Complexity of Parallel Merge-sort with Parallel Merge?

First, the new code - Input $A[p..r]$ - Output $B[s..s+r-p]$

```
$Par-Merge-Sort(A,p,r,B,s)$
    $n = r - p + 1$
    if $(n == 1)$
            $B[s] = A[p]$
    else let $T[1..n]$ be a new array
            $q = \lfloor (p+r)/2 \rfloor$
            $q = q - p + 1$
            spawn $Par-Merge-Sort(A,p,q,T,1)$
            $Par-Merge-Sort(A,q+1,r,T,q'+1)$
            sync
            $Par-Merge(T,1,q',q'+1,n,B,s)$
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        $q=\left\lfloor {}^{(p+r)}/_{2}\right\rfloor$
        $q=q-p+1$
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Then, What is the amount of Parallelism of Parallel Merge-sort with Parallel Merge?

Work

We can use the worst work in the parallel to generate the recursion:

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\begin{aligned}
T_1^{PMS}(n) &= 2T_1^{PMS}\left(\frac{n}{2}\right) + T_1^{PM}(n) \\
&= 2T_1^{PMS}\left(\frac{n}{2}\right) + \Theta(n) \\
&= \Theta(n \log n) \text{ Case 2 of the MT}
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We get the following recursion for the span by taking in account that lines 7 and 8 of parallel merge sort run in parallel:

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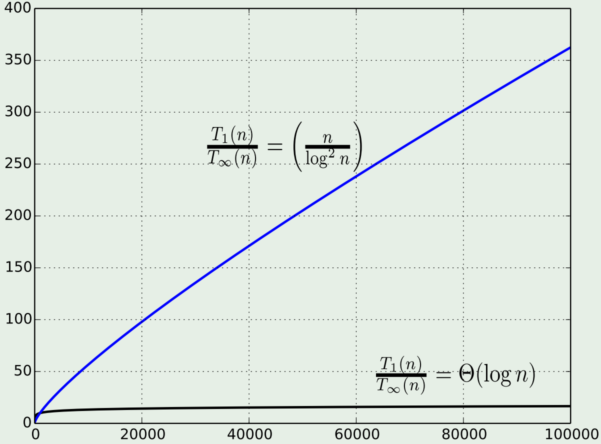
Parallelism

$$\frac{T_1(n)}{T_\infty(n)} = \Theta\left(\frac{n}{\log^2 n}\right)$$

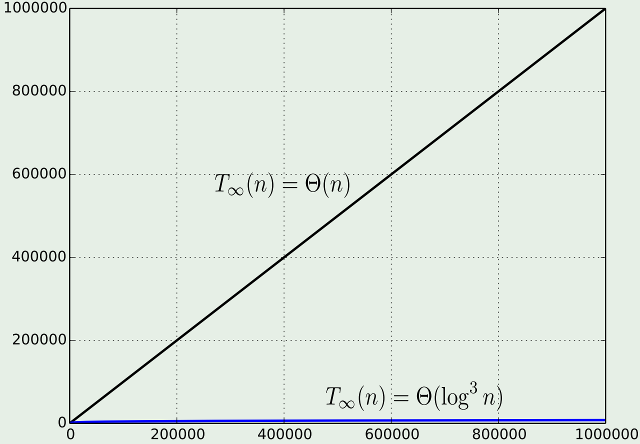
Plotting both Parallelisms

We get the incredible difference between both algorithm



Plotting the T_∞

We get the incredible difference when running both algorithms with an infinite number of processors!!!



Outline

Exercises

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- 27.1-2
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- 27.2-1
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