Analysis of Algorithms

All-Pairs Shortest Path

Andres Mendez-Vazquez

November 11, 2015

Outline

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Problem

Definition

- Given u and v, find the shortest path.
- Now, what if you want **ALL PAIRS**!!!
- Use as a source all the elements in V.
- Clearly!!! you can fall back to the old algorithms!!!

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What can we use?

Use Dijkstra's $|V|$ times!!!

- **If all the weights are non-negative.**

- Then, we have $O(V^2 E)$.
- Which is equal $O(V^4)$ in the case of $E = O(V^2)$.

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Problems

- Computer Network Systems.
- Aircraft Networks (e.g. flying time, fares).
- Railroad network tables of distances between all pairs of cites for a road atlas.
- Etc.

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As many things in the history of analysis of algorithms the all-pairs shortest path has a long history.

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Outline

Assumptions Matrix Representation

Matrix Representation of a Graph

For this, we have that each weight in the matrix has the following values

$$w_{ij} = \begin{cases} 0 & \text{if } i = j \\ w(i,j) & \text{if } i \neq j \text{ and } (i,j) \in E \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E \end{cases}$$

Then, we have $W = \begin{pmatrix} w_{11} & w_{22} & \dots & w_{1k-1} & w_{1n} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ w_{n1} & w_{n2} & \dots & w_{nn-1} & w_{nn} \end{pmatrix}$

Important

- There are not negative weight cycles.

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At the end of the algorithm will generate the following matrix

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Each entry $d_{ij} = \delta(i, j)$.

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Structure of a Shortest Path

Consider Lemma 24.1

Given a weighted, directed graph $G = (V, E)$ with $p =< v_1, v_2, ..., v_k >$ be a SP from v_1 to v_k. Then,

- $p_{ij} =< v_i, v_{i+1}, ..., v_j >$ is a Shortest Path (SP) from v_i to v_j, where $1 \leq i \leq j \leq k$.

We can do the following

- Consider the shortest path p from vertex i and j, p **contains at most** m **edges.**

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We can do the following

- Consider the shortest path p from vertex i and j, p **contains at most** m **edges**.
- Then, we can use the Corollary to make a decomposition

$$i \overset{p'}{\leadsto} k \to j \Longrightarrow \delta(i, j) = \delta(i, k) + w_{kj}$$

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Structure of a Shortest Path

Idea of Using Matrix Multiplication

- We define the following concept based in the decomposition Corollary!!!
- $l_{ij}^{(m)}$ =minimum weight of any path from i to j, it contains at most m edges i.e.

$$l_{ij}^{(m)} \text{ could be } \min_{k}\left\{ l_{ik}^{(m-1)} + w_{kj} \right\}$$

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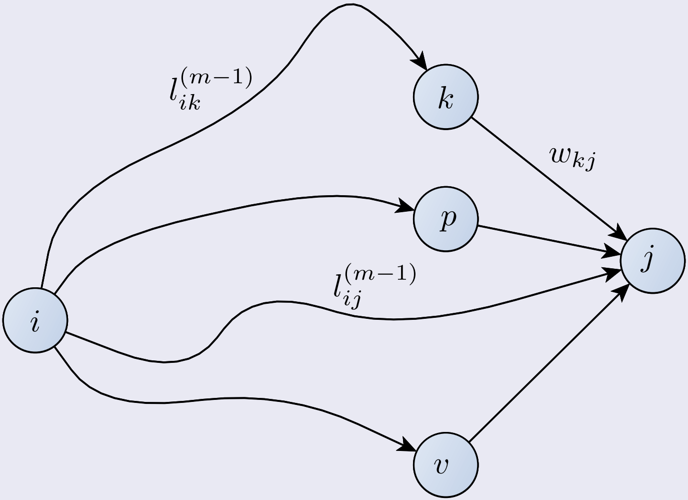
Graphical Interpretation

Looking for the Shortest Path



Outline

Recursive Solution

Thus, we have that for paths with ZERO edges

$$l_{ij}^{(0)} = \begin{cases} 0 & \textbf{if } i = j \\ \infty & \textbf{if } i \neq j \end{cases}$$

Recursion Our Great Friend

- Consider the previous definition and decomposition. Thus

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$$\begin{aligned} l_{ij}^{(m)} &= \min\left(l_{ij}^{(m-1)}, \min_{1 \leq k \leq n} \left\{ l_{ik}^{(m-1)} + w_{kj} \right\} \right) \\ &= \min_{1 \leq k \leq n} \left\{ l_{ik}^{(m-1)} + w_{kj} \right\} \end{aligned}$$

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Recursive Solution

Why? A simple notation problem

$$l_{ij}^{(m)} = l_{ij}^{(m-1)} + 0 = l_{ij}^{(m-1)} + w_{jj}$$

Outline

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Transforming it to a iterative one

What is $\delta(i,j)$?

- If you do not have negative-weight cycles, and $\delta(i,j) < \infty$.
- Then, the shortest path from vertex i to j has at most $n-1$ edges

$$\delta(i,j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = l_{ij}^{(n+2)} = \dots$$

- Then, we can compute first $L^{(1)}$ then compute $L^{(2)}$ all the way to $L^{(n-1)}$ which contains the actual shortest paths.

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Algorithm

Code

Extended-Shortest-Path(L, W)

1. $n = L.rows$
2. **let** $L' = \left(l'_{ij}\right)$ **be a new** $n \times n$
3. **for** $i = 1$ **to** n
4. **for** $j = 1$ **to** n
5. $l'_{ij} = \infty$
6. **for** $k = 1$ **to** n
7. $l'_{ij} = \min\left(l'_{ij}, l_{ik} + w_{kj}\right)$
8. **return** L'

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Complexity

If $|V| == n$ we have that $\Theta\left(V^3\right)$.

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Look Alike Matrix Multiplication Operations

Mapping That Can be Thought

- $L \Longrightarrow A$
- $W \Longrightarrow B$
- $L' \Longrightarrow C$
- $\min \Longrightarrow +$
- $+ \Longrightarrow \cdot$
- $\infty \Longrightarrow 0$

Look Alike Matrix Multiplication Operations

Using the previous notation, we can rewrite our previous algorithm as

Square-Matrix-Multiply(A, B)

```
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

Complexity

Thus

The complexity of the **Extended-Shortest-Path** is equal to $O\left(n^3\right)$

Using the Analogy

Returning to the all-pairs shortest-paths problem

It is possible to compute the shortest path by extending such a path edge by edge.

Therefore

If we denote $A \cdot B$ as the "product" of the **Extended-Shortest-Path**

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Using the Analogy

We have that

$$
\begin{aligned}
L^{(1)} = &\quad L^{(0)} \cdot W = &\quad W \\
L^{(2)} = &\quad L^{(1)} \cdot W = &\quad W^{2} \\
&\quad \vdots & \\
L^{(n-1)} = &\quad L^{(n-2)} \cdot W = &\quad W^{n-1}
\end{aligned}
$$

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The Final Algorithm

We have that

Slow-All-Pairs-Shortest-Paths(W **)**

1. $n \leftarrow W.rows$
2. $L^{(1)} \leftarrow W$
3. **for** $m = 2$ **to** $n - 1$
4. $\quad L^{(m)} \leftarrow$ **EXTEND-SHORTEST-PATHS** $\left(L^{(m-1)}, W\right)$
5. **return** $L^{(n-1)}$

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With Complexity

Complexity

$$O\left(V^4\right) \tag{1}$$

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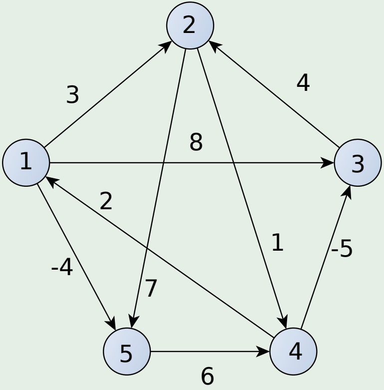
Example

We have the following



	1	2	3	4	5
1	0	2	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$$L^{(1)} = L^{(0)}W$$

Example

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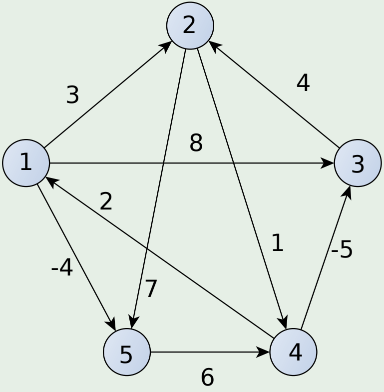
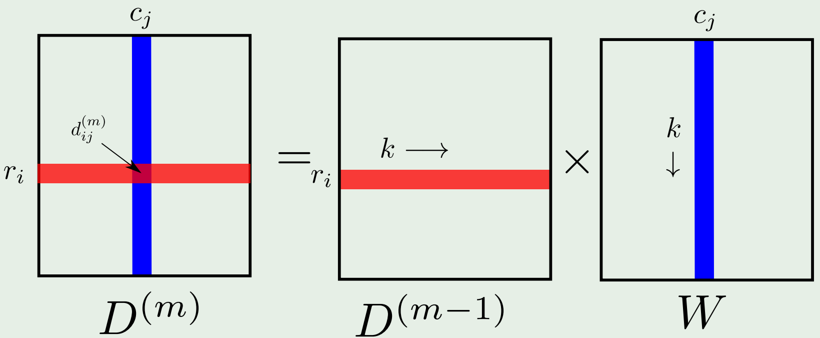


	1	2	3	4	5
1	0	2	8	2	-4
2	3	0	-4	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	8	∞	1	6	0

$$L^{(2)} = L^{(1)}W$$

Here, we use the analogy of matrix multiplication

$D^1 W$



Thus, the update of an element l_{ij}

Example

$$
\begin{aligned}
l_{14}^{(2)} &= \min\left\{ \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \end{pmatrix} + \begin{pmatrix} \infty \\ 1 \\ \infty \\ 0 \\ 6 \end{pmatrix} \right\} \\
&= \min\begin{pmatrix} \infty & 4 & \infty & \infty & 2 \end{pmatrix} \\
&= 2
\end{aligned}
$$

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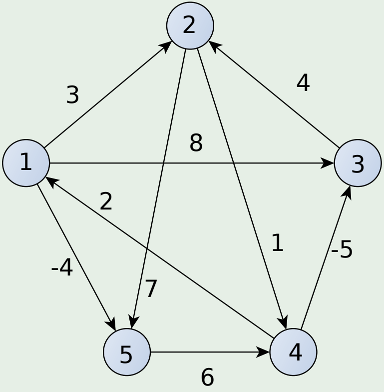

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$L^{(3)} = L^{(2)}W$

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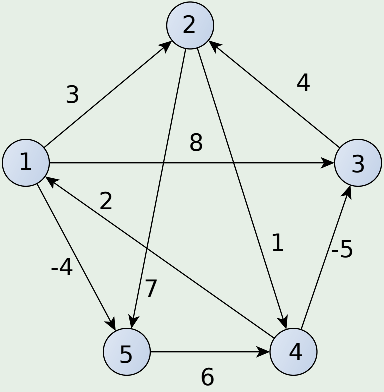

	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

$$L^{(4)} = L^{(3)}W$$

Outline

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Recall the following

We are interested only

In matrix $L^{(n-1)}$

In addition

Remember, we do not have negative weight cycles!!!

Therefore, given the equation

$$\delta(i,j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n)} = \ldots \tag{2}$$

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Thus

It implies

$$L^{(m)} = L^{(n-1)} \tag{3}$$

For all

$$m \geq n - 1 \tag{4}$$

Thus

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$$L^{(m)} = L^{(n-1)} \tag{3}$$

For all

$$m \geq n - 1 \tag{4}$$

Something Faster

We want something faster!!! Observation!!!

$$L^{(1)} = \quad W$$

$$2^{\lceil \lg(n-1) \rceil} \geq n-1 \Longrightarrow L^{\left(2^{\lceil \lg(n-1) \rceil}\right)} = L^{(n-1)}$$

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L^{(1)} &= W \\
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L^{(8)} &= W^4 \cdot W^4 = W^8 \\
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The Faster Algorithm

Complexity of the Previous Algorithm

Slow-All-Pairs-Shortest-Paths(W **)**

$n \leftarrow W.rows$

$L^{(1)} \leftarrow W$

$m \leftarrow 1$

while $m < n - 1$

    $L^{(2m)} \leftarrow$ **EXTEND-SHORTEST-PATHS** $\left(L^{(m)}, L^{(m)}\right)$

    $m \leftarrow 2m$

return $L^{(m)}$

Complexity

If $n = |V|$ we have that $O\left(V^3 \lg V\right)$.

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Outline

The Shortest Path Structure

Intermediate Vertex

For a path $p = \langle v_1, v_2, ..., v_l \rangle$, an **intermediate vertex** is any vertex of p other than v_1 or v_l.

Define

$d_{ij}^{(k)}$ =weight of a shortest path between i and j with all intermediate vertices are in the set $\{1, 2, ..., k\}$.

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The Recursive Idea

Simply look at the following cases cases

- **Case I** k is not an intermediate vertex, then a shortest path from i to j with all intermediate vertices $\{1, ..., k-1\}$ is a shortest path from i to j with intermediate vertices $\{1, ..., k\}$.

$$\Longrightarrow d_{ij}^{(k)} = d_{ij}^{(k-1)}$$

- **Case II** if k is an intermediate vertice. Then, $i \overset{p1}{\rightsquigarrow} k \overset{p2}{\rightsquigarrow} j$ and we can make the following statements using Lemma 24.1:
  - p_1 is a shortest path from i to k with all intermediate vertices in the set $\{1, ..., k-1\}$.
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The Graphical Idea

Consider

All possible intermediate vertices in $\{1, 2, ..., k\}$

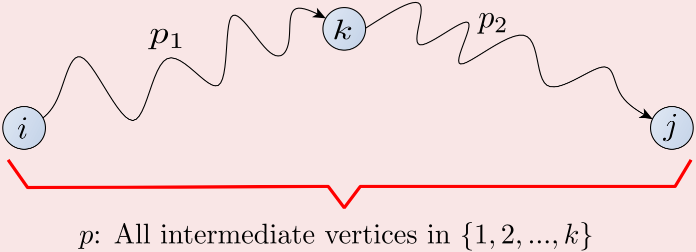

Figure: The Recursive Idea

The Recursive Solution

The Recursion

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0 \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if } k \geq 1 \end{cases}$$

Final answer when $k = n$

We recursively calculate $D^{(n)} = \left(d_{ij}^{(n)}\right)$ or $d_{ij}^{(n)} = \delta(i,j)$ for all $i, j \in V$.

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Thus, we have the following

Recursive Version

Recursive-Floyd-Warshall(W)

1. $D^{(n)}$ the $n \times n$ matrix
2. for $i = 1$ to n
3. for $j = 1$ to n
4. $D^{(n)}[i, j] =$Recursive-Part(i, j, n, W)
5. return $D^{(n)}$

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The Recursive-Part

Recursive-Part(i, j, k, W)

1. if $k = 0$
2. return $W[i, j]$
3. if $k \geq 1$
4. t_1 =Recursive-Part$(i, j, k-1, W)$
5. t_2 =Recursive-Part$(i, k, k-1, W)$+...
6. Recursive-Part$(k, j, k-1, W)$
7. if $t_1 \leq t_2$
8. return t_1
9. else
10. return t_2

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Recursive-Part(i, j, k, W)

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if k = 0
        return W[i, j]
if k ≥ 1
        t_1 = Recursive-Part(i, j, k − 1, W)
        t_2 = Recursive-Part(i, k, k − 1, W)+...
                    Recursive-Part(k, j, k − 1, W)
        if t_1 ≤ t_2
                return t_1
        else
                return t_2
```

Outline

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Now

We want to use a storage to eliminate the recursion

For this, we are going to use two matrices

1. $D^{(k-1)}$ the previous matrix.
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For this, we have the predecessor matrix Π

Actually, we want to compute a sequence of matrices

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$$\Pi = \Pi^{(n)}$$

What are the elements in $\Pi^{(k)}$

Each element in the matrix is as follow

$\pi_{ij}^{(k)}$ = the predecessor of vertex j on a shortest path from vertex i with all intermediate vertices in the set $\{1, 2, ..., k\}$

Thus, we have that

$$\pi_{ij}^{(0)} = \begin{cases} NULL & \text{if } i = j \text{ or } w_{ij} = \infty \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty \end{cases}$$

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We have the following

For $k \geq 1$, if we take the path $i \rightsquigarrow k \rightsquigarrow j$ where $k \neq j$.

Then, if $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

For the predecessor of j, we chose k on a shortest path from k with all intermediate vertices in the set $\{1, 2, ..., k-1\}$

Otherwise, if $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

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Formally

We have then

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases}$$

Outline

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Final Iterative Version of Floyd-Warshall (Correction by Diego - Class Tec 2015)

Floyd-Warshall(W **)**

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n = W.rows
D^(0) = W
for k = 1 to n − 1
    let D^(k) = (d_ij^(k)) be a new n × n matrix
    let Π^(k) be a new predecessor n × n matrix
    ▷ Given each k, we update using D^(k−1)
    for i = 1 to n
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11. **else**
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Initialization of variables n and $D^{(0)}$

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This is done to go through all the possible combinations of i's and j's

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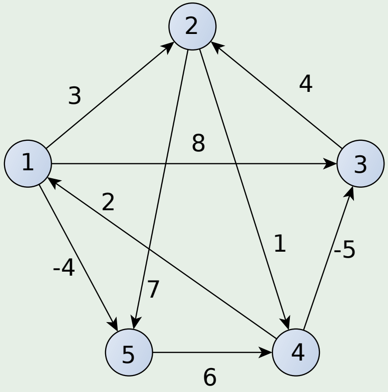
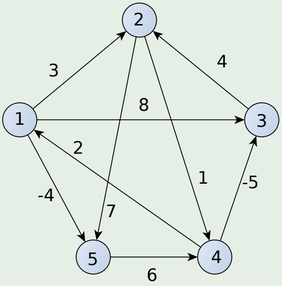
Example

Graph



Example

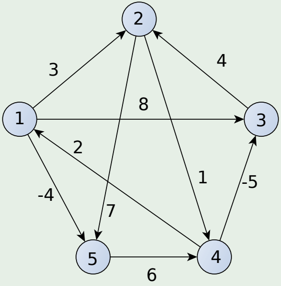
$D^{(0)}$ and $\Pi^{(0)}$



$$D^{(0)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad \Pi^{(0)} = \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

Example

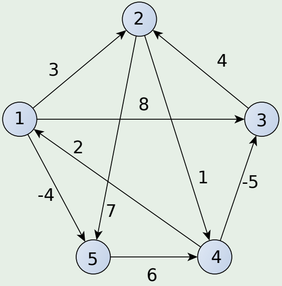
$D^{(1)}$ and $\Pi^{(1)}$



$$D^{(1)} = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad \Pi^{(1)} = \begin{bmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

Example

$D^{(2)}$ and $\Pi^{(2)}$



$$D^{(2)} = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad \Pi^{(2)} = \begin{bmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{bmatrix}$$

Example

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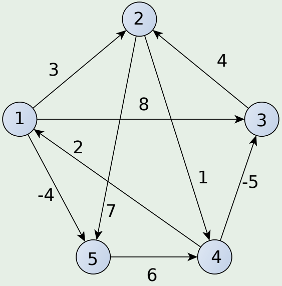

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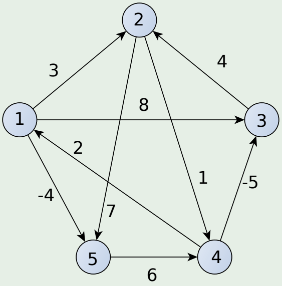

$$D^{(4)} = \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \quad \Pi^{(4)} = \begin{bmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{bmatrix}$$

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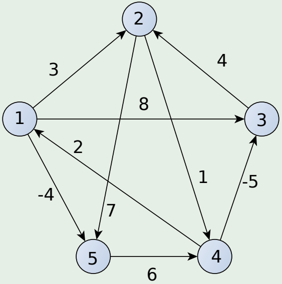

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Because the comparison in line 8 takes $O(1)$

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Outline

Cinvestav

Johnson's Algorithm

Observations

- Used to find all pairs in a sparse graphs by using Dijkstra's algorithm.
- It uses a re-weighting function to obtain positive edges from negative edges to deal with them.
- It can deal with the negative weight cycles.

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- It uses something to deal with the negative weight cycles.
  - Could be a Bellman-Ford detector as before?
- Maybe, we need to transform the weights in order to use them.

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Proving Properties of Re-Weigthing

Lemma 25.1

Given a weighted, directed graph $G = (D, V)$ with weight function $w : E \rightarrow \mathbb{R}$, let $h : V \rightarrow \mathbb{R}$ be any function mapping vertices to real numbers. For each edge $(u, v) \in E$, define

$$\widehat{w}(u, v) = w(u, v) + h(u) - h(v)$$

Let $p = \langle v_0, v_1, ..., v_k \rangle$ be any path from vertex 0 to vertex k. Then:

1. p is a shortest path from 0 to k with weight function w if and only if it is a shortest path with weight function $\widehat{w}$. That is $w(p) = \delta(v_0, v_k)$ if and only if $\widehat{w}(p) = \widehat{\delta}(v_0, v_k)$.
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Using This Lemma

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Such that $w(u,v) + h(u) - h(v) \geq 0$.

- $V' = V \cup \{s\}$, where s is a new vertex.
- $E' = E \cup \{(s,v) | v \in V\}$.
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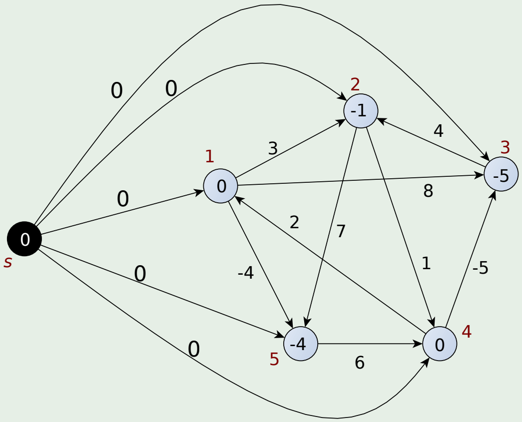
Example

Graph G' with original weight function w with new source s and $h(v) = \delta(s, v)$ at each vertex



Proof of Claim

Claim

$$w(u, v) + h(u) - h(v) \geq 0 \quad (5)$$

By Triangle Inequality

- $\delta(s, v) \leq \delta(s, u) + w(u, v)$

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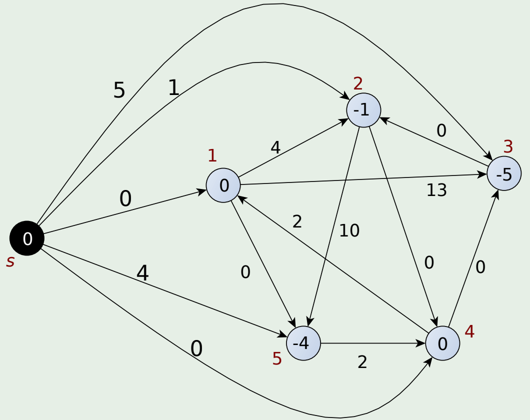
The new Graph G after re-weighting G'



Final Algorithm

Pseudo-Code

Compute G', where: $G'.V = G.E \cup \{(s,v) \,|\, v \in G.V\}$ and $w(s,v) = 0$ for all $v \in G.V$

If Bellman-Ford$(G', w, s) == FALSE$

    print "Graphs contains a Neg-Weight Cycle"

else for each vertex $v \in G'.V$

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    for each edge $(u,v) \in G'.E$

        $\widehat{w}(u,v) = w(u,v) + h(u) - h(v)$

    Let $D = (d_{uv})$ be a new $n \times n$ matrix

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Final Algorithm

Pseudo-Code

Compute G', where: $G'.V = G.E \cup \{(s, v) \mid v \in G.V\}$ and $w(s, v) = 0$ for all $v \in G.V$

If Bellman-Ford$(G', w, s) == FALSE$

    print "Graphs contains a Neg-Weight Cycle"

else for each vertex $v \in G'.V$

        set $h(v) = v.d$ computed by Bellman-Ford

    for each edge $(u, v) \in G'.E$

        $\widehat{w}(u, v) = w(u, v) + h(u) - h(v)$

    Let $D = (d_{uv})$ be a new $n \times n$ matrix

    for each vertex $u \in G.V$

        run Dijkstra$(G, \widehat{w}, u)$ to compute $\widehat{\delta}(u, v)$ for all $v \in G.V$

        for each vertex $v \in G.V$

            $d_{uv} = \widehat{\delta}(u, v) + h(v) - h(u)$

    return D

Complexity

The Final Complexity

- Times:
  - $\Theta(V+E)$ to compute G'
  - $O(VE)$ to run Bellman-Ford
  - $\Theta(E)$ to compute $\widehat{w}$
  - $O\left(V^2 \lg V + VE\right)$ to run Dijkstra's algorithm $|V|$ time using Fibonacci Heaps
  - $O\left(V^2\right)$ to compute D matrix
- Total : $O(V^2 \lg V + VE)$
- If $E = O\left(V^2\right) \implies O\left(V^3\right)$

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Outline

Excercises

- 25.1-4
- 25.1-8
- 25.1-9
- 25.2-4
- 25.2-6
- 25.2-9
- 25.3-3