Analysis of Algorithms

Single Source Shortest Path

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Outline

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Introduction

Problem description

- Given a single source vertex in a weighted, directed graph.
- We want to compute a shortest path for each possible destination (Similar to BFS).

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Thus

The algorithm will compute a shortest path tree (again, similar to BFS).

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Single destination shortest paths problem

Find a shortest path to a given destination vertex t from each vertex.

- By reversing the direction of each edge in the graph, we can reduce this problem to a single source problem.

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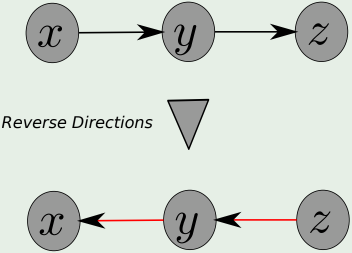

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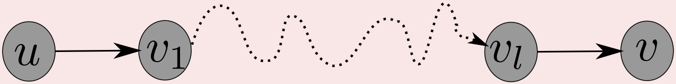
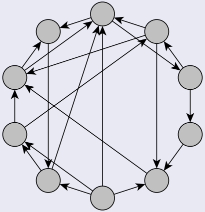


Similar Problems

All pairs shortest paths problem

Find a shortest path from u to v for every pair of vertices u and v.



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Optimal Substructure Property

Lemma 24.1

Given a weighted, directed graph $G = (V, E)$ with $p =< v_1, v_2, ..., v_k >$ be a **Shortest Path** from v_1 to v_k. Then,

- $p_{ij} =< v_i, v_{i+1}, ..., v_j >$ is a **Shortest Path** from v_i to v_j, where $1 \leq i \leq j \leq k$.

We have then

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Corollary

Let p be a Shortest Path from s to v, where $p = s \overset{p_1}{\rightsquigarrow} u \rightarrow v = p_1 \cup \{(u, v)\}$. Then $\delta(s, v) = \delta(s, u) + w(u, v)$.

The Lower Bound Between Nodes

Lemma 24.10

Let $s \in V$. For all edges $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.

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The Predecessor Graph

This will facilitate the proof of several concepts

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Predecessor Graph

Representing shortest paths

For this we use the **predecessor subgraph**

- It is defined slightly differently from that on Breadth-First-Search

- $V_{\pi} = \{v \in V | v.\pi \neq NIL\} \cup \{s\}$
- $E_{\pi} = \{(v.\pi, v) | v \in V_{\pi} - \{s\}\}$

- The edges in E_{π} are called tree edges

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- The predecessor subgraph G_π forms a depth first forest composed of several depth first trees.
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The Relaxation Concept

We are going to use certain functions for all the algorithms

- Initialize
  - Here, the basic variables of the nodes in a graph will be initialized
    - $v.d$ = the distance from the source s.
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Initialize and Relaxation

The Algorithms keep track of $v.d$, $v.\pi$. It is initialized as follows

```
Initialize($G, s$)
for each $v \in V[G]$
        $v.d = \infty$
        $v.\pi = NIL$
$s.d = 0$
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These values are changed when an edge (u, v) is relaxed.

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Relax($u, v, w$)
if $v.d > u.d + w(u, v)$
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Bellman-Ford can have negative weight edges. It will "detect" reachable negative weight cycles.

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Bellman-Ford($G, s, w$)
Initialize($G, s$)
for $i = 1$ to $|V[G]| - 1$
    for each $(u, v)$ to $E[G]$
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for each $(u, v)$ to $E[G]$
    if $v.d > u.d + w(u, v)$
        return false
return true
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- After i iterations of relaxing an all (u, v), if the shortest path to v has i edges, then $v.d = \delta(s, v)$.
- If there is no path from s to v, then $v.d = \delta(s, v) = \infty$ is an invariant.

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Lemma 24.10 (Triangle inequality)

Let $G=(V,E)$ be a weighted, directed graph with weight function $w:E\rightarrow\mathbb{R}$ and source vertex s. Then, for all edges $(u,v)\in E$, we have:

$$\delta(s,v)\leq\delta(s,u)+w(u,v) \tag{1}$$

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Proof

1. Suppose that p is a shortest path from source s to vertex v.
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- Then, we have that $v.d \geq \delta(s, v)$ $\forall v \in V[G]$, and this invariant is maintained over any sequence of relaxation steps on the edges of G.
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Loop Invariance

The Proof can be done by induction over the number of relaxation steps and the loop invariance:

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For the Basis

$v.d \geq \delta\left(s,v\right)$ is true after initialization, since:

- $v.d = \infty$ making $v.d \geq \delta\left(s,v\right)$ for all $v \in V - \{s\}$.
- For s, $s.d = 0 \geq \delta\left(s,s\right)$.

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By the inductive hypothesis, we have that $x.d \geq \delta(s,x)$ for all $x \in V$ prior to relaxation.

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If you call $Relax(u, v, w)$, it may change $v.d$

$$\begin{aligned} v.d &= u.d + w(u, v) \\ &\geq \delta(s, u) + w(u, v) \text{ by inductive hypothesis} \\ &\geq \delta\left(s, v\right) \text{ by the triangle inequality} \end{aligned}$$

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$$\begin{aligned} v.d &= u.d + w(u, v) \\ &\geq \delta(s, u) + w(u, v) \text{ by inductive hypothesis} \\ &\geq \delta\left(s, v\right) \text{ by the triangle inequality} \end{aligned}$$

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Properties of Relaxation

Proof of lemma 24.11 cont...

- To proof that the value $v.d$ never changes once $v.d = \delta\left(s, v\right)$:
    - Note the following: Once $v.d = \delta\left(s, v\right)$, it cannot decrease because $v.d \geq \delta\left(s, v\right)$ and Relaxation never increases d.

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Next, we have

Corollary 24.12 (No-path property)

If there is no path from s to v, then $v.d = \delta(s, v) = \infty$ is an invariant.

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By the upper-bound property, we always have $\infty = \delta(s, v) \leq v.d$. Then, $v.d = \infty$.

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More Lemmas

Lemma 24.13

Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \rightarrow \mathbb{R}$. Then, immediately after relaxing edge (u, v) by calling $Relax(u, v, w)$ we have $v.d \leq u.d + w(u, v)$.

Proof

First

If, just prior to relaxing edge (u, v),

- Case 1: if we have that $v.d > u.d + w(u, v)$
    - Then, $v.d = u.d + w(u, v)$ after relaxation.

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If $v.d \leq u.d + w(u, v)$ just before relaxation, then:

- neither $u.d$ nor $v.d$ changes

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More Lemmas

Lemma 24.14 (Convergence property)

- Let p be a shortest path from s to v, where $p = s \overset{p_1}{\rightsquigarrow} u \rightarrow v = p_1 \cup \{(u,v)\}$.
- If $u.d = \delta(s,u)$ holds at any time prior to calling $Relax(u,v,w)$, then $v.d = \delta(s,v)$ holds at all times after the call.

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Proof

Thus , after relaxing (u, v)

$$v.d \leq u.d + w(u,v) \text{ by lemma 24.13}$$

$$= \delta(s,u) + w(u,v)$$

$$= \delta(s,v) \text{ by corollary of lemma 24.1}$$

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By lemma 24.11, $v.d \geq \delta(s,v)$, so $v.d = \delta(s,v)$.

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Predecessor Subgraph for Bellman

Lemma 24.16

Assume a given graph G that has no negative weight cycles reachable from s. Then, after the initialization, the predecessor subgraph G_π is always a tree with root s, and any sequence of relaxations steps on edges of G maintains this property as an invariant.

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Proof of G_π is acyclic

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Suppose there exist a cycle $c =< v_0, v_1, ..., v_k >$, where $v_0 = v_k$. We have $v_i.\pi = v_{i-1}$ for $i = 1, 2, ..., k$.

Second

Assume relaxation of (v_{k-1}, v_k) created the cycle. We are going to show that the cycle has a negative weight.

We claim that

The cycle must be reachable from s (Why?)

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Each vertex on the cycle has a non-NIL predecessor, and so each vertex on it was assigned a finite shortest-path estimate when it was assigned its non-NIL value.

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By the upper-bound property, each vertex on the cycle has a finite shortest-path weight.

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Proof

Before call to $Relax(v_{k-1}, v_k, w)$:

$$v_i.\pi = v_{i-1} \text{ for } i = 1, ..., k-1. \tag{2}$$

Thus

This Implies $v_i.d$ was last updated by

$$v_i.d = v_{i-1}.d + w(v_{i-1}, v_i) \tag{3}$$

for $i = 1, ..., k-1$ (Because $Relax$ updates π).

This implies

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Because $v_k.\pi$ is changed by call $Relax$ (Immediately before), $v_k.d > v_{k-1}.d + w(v_{k-1}, v_k)$, we have that:

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We have finally that

Because $\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d$, we have that $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$, i.e., a negative weight cycle!!!

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- $v_i.d \geq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, ..., k-1$ because when $Relax(v_{i-1}, v_i, w)$ was called, there was an equality, and $v_{i-1}.d$ may have gotten smaller by further calls to $Relax$.
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Proof of existence of a unique path from source s

Let G_π be the predecessor subgraph.

- So, for any v in V_π, the graph G_π contains at least one path from s to v.
- Assume now that you have two paths:
- This can only be possible if for two nodes x and $y \Rightarrow x \neq y$, but $z.\pi = x = y$!!!
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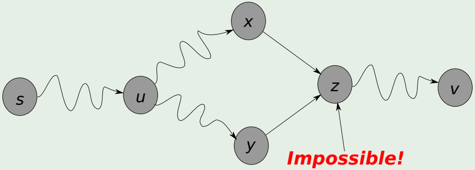

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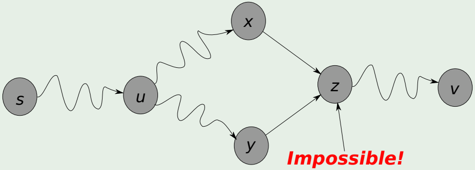

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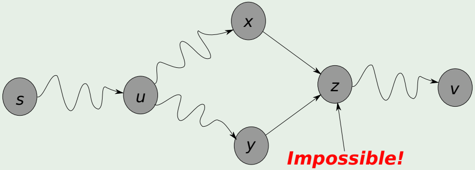

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Same conditions as before. It calls Initialize and repeatedly calls Relax until $v.d = \delta(s, v)$ for all v in V. Then G_{π} is a shortest path tree rooted at s.

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Let $p =< v_0, v_1, ..., v_k >$, where $v_0 = s$ and $v_k = v$. Thus, we have $v_i.d = \delta(s, v_i)$.

And reasoning as before

$$v_i.d \geq v_{i-1}.d + w(v_{i-1}, v_i) \tag{5}$$

This implies that

$$w(v_{i-1}, v_i) \leq \delta(s, v_i) - \delta(s, v_{i-1}) \tag{6}$$

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\begin{aligned}
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Outline

Again the Bellman-Ford Algorithm

Bellman-Ford can have negative weight edges. It will "detect" reachable negative weight cycles.

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Bellman-Ford($G, s, w$)
Initialize($G, s$)
for $i = 1$ to $|V[G]| - 1$
    for each $(u, v)$ to $E[G]$
        Relax($u, v, w$)◁ The Decision Part of the Dynamic Programming for $u.d$ and $u.\pi$.
for each $(u, v)$ to $E[G]$
    if $v.d > u.d + w(u, v)$
        return false
return true
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Observation

If Bellman-Ford has not converged after $V(G) - 1$ iterations, then there cannot be a shortest path tree, so there must be a negative weight cycle.

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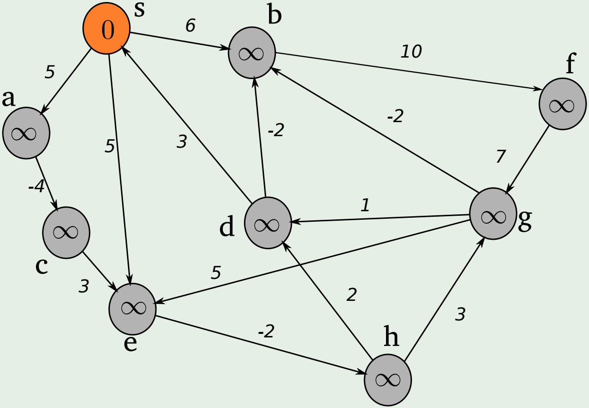
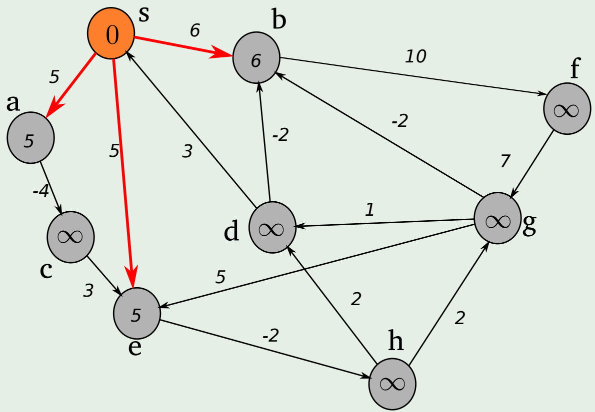
Example

Red Arrows are the representation of $v.\pi$



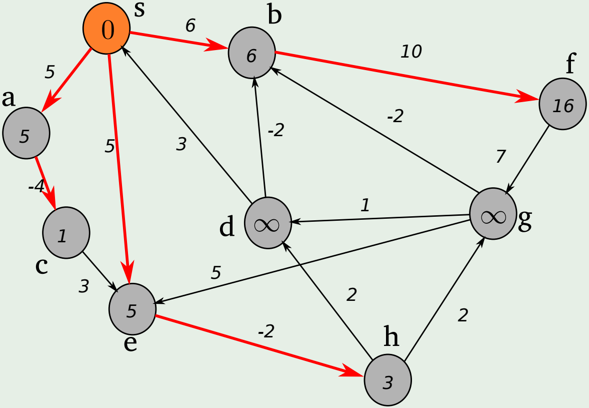
Example

Here, whenever we have $v.d = \infty$ and $v.u = \infty$ no change is done



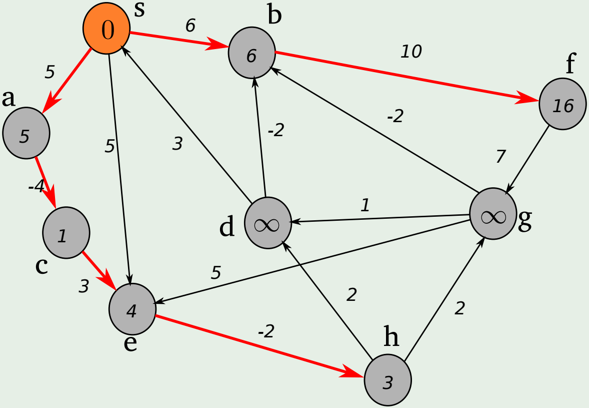
Example

Here, we keep updating in the second iteration



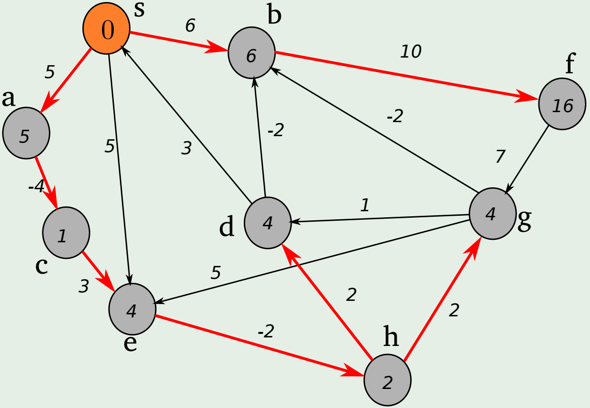
Example

Here, during it. we notice that e can be updated for a better value



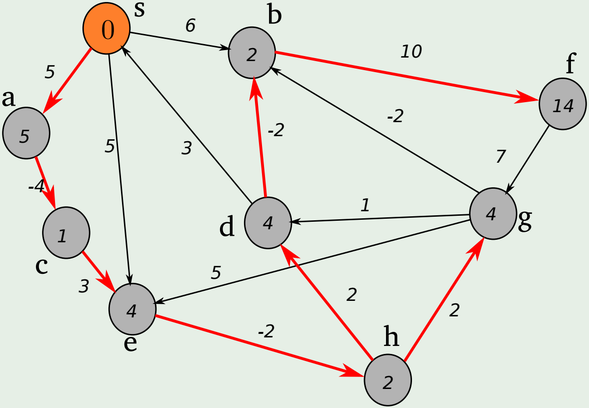
Example

Here, we keep updating in third iteration and d and g also is updated



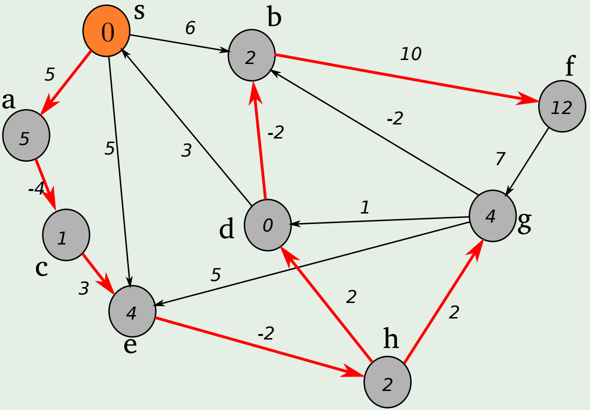
Example

Here, we keep updating in fourth iteration



Example

Here, f is updated during this iteration



Outline

$v.d == \delta(s, v)$ upon termination

Lemma 24.2

Assuming no negative weight cycles reachable from s, $v.d == \delta(s, v)$ holds upon termination for all vertices v reachable from s.

Proof

Consider a shortest path p, where $p =< v_0, v_1, ..., v_k >$, where $v_0 = s$ and $v_k = v$.

We know the following

Shortest paths are simple, p has at most $|V| - 1$, thus we have that $k \leq |V| - 1$.

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Claim: $v_i.d = \delta(s, v_i)$ holds after the ith pass over edges.

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Each of the $|V| - 1$ iterations of the **for** loop (Lines 2-4) relaxes all edges in E.

- By lemma 24.11, once $v_i.d = \delta(s, v_i)$ holds, it continues to hold.

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Proof follows by induction on i

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Finding a path between s and v

Corollary 24.3

Let $G = (V, E)$ be a weighted, directed graph with source vertex s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, for each vertex $v \in V$, there is a path from s to v if and only if Bellman-Ford terminates with $v.d < \infty$ when it is run on G

Proof

Left to you...

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Outline

Correctness of Bellman-Ford

Claim: The Algorithm returns the correct value

Part of Theorem 24.4. Other parts of the theorem follow easily from earlier results.

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Upon termination, we have for all (u, v):

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by lemma 24.2 (last slide) if v is reachable or $v.d = \delta(s, v) = \infty$ otherwise.

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Correctness of Bellman-Ford

Then, we have that

$$
\begin{aligned}
v.d = \delta(s,v) \\
&\leq \delta(s,u) + w(u,v) \\
&\leq u.d + w(u,v)
\end{aligned}
$$

Remember:

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for each $(u,v)$ to $E[G]$
    if $v.d > u.d + w(u,v)$
        return false
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5. for each $(u,v)$ to $E[G]$
6.     if $v.d > u.d + w(u,v)$
7.         return false
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Correctness of Bellman-Ford

Case 2: There exists a reachable negative weight cycle $c =< v_0, v_1, ..., v_k >$, where $v_0 = v_k$.

Then, we have:

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0. \tag{7}$$

Suppose algorithm returns *true*

Then $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1, ..., k$ because Relax did not change any $v_i.d$.

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Thus

$$\begin{aligned}\sum_{i=1}^{k} v_i.d &\leq \sum_{i=1}^{k} \left(v_{i-1}.d + w(v_{i-1}, v_i)\right)\\ &= \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)\end{aligned}$$

Since $v_0 = v_k$

Each vertex in c appears exactly once in each of the summations, $\sum_{i=1}^{k} v_i.d$ and $\sum_{i=1}^{k} v_{i-1}.d$, thus

$$\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d \qquad (8)$$

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By Corollary 24.3

$v_i.d$ is finite for $i = 1, 2, ..., k$, thus

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This contradicts (Eq. 7). Thus, algorithm returns *false*.

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Outline

1. Introduction
   - Introduction and Similar Problems
2. General Results
   - Optimal Substructure Properties
   - Predecessor Graph
   - The Relaxation Concept
   - The Bellman-Ford Algorithm
   - Properties of Relaxation
3. Bellman-Ford Algorithm
   - Predecessor Subgraph for Bellman
   - Shortest Path for Bellman
   - Example
   - Bellman-Ford finds the Shortest Path
   - Correctness of Bellman-Ford
4. Directed Acyclic Graphs (DAG)
   - Relaxing Edges
   - Example
5. Dijkstra's Algorithm
   - Dijkstra's Algorithm: A Greedy Method
   - Example
   - Correctness Dijkstra's algorithm
   - Complexity of Dijkstra's Algorithm
6. Exercises

Another Example

Something Notable

By relaxing the edges of a weighted DAG $G = (V, E)$ according to a topological sort of its vertices, we can compute shortest paths from a single source in time.

Why?

Shortest paths are always well defined in a DAG, since even if there are negative-weight edges, no negative-weight cycles can exist.

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Single-source Shortest Paths in Directed Acyclic Graphs

In a DAG, we can do the following (Complexity $\Theta\left(V+E\right)$)

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DAG -SHORTEST-PATHS$(G, w, s)$
    Topological sort vertices in G
    Initialize$(G, s)$
    for each $u$ in $V\left[G\right]$ in topological sorted order
        for each $v$ to $Adj\left[u\right]$
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It is based in the following theorem

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If a weighted, directed graph $G = (V, E)$ has source vertex s and no cycles, then at the termination of the DAG-SHORTEST-PATHS procedure, $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest path.

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We have that

1. Line 1 takes $\Theta\left(V+E\right)$.
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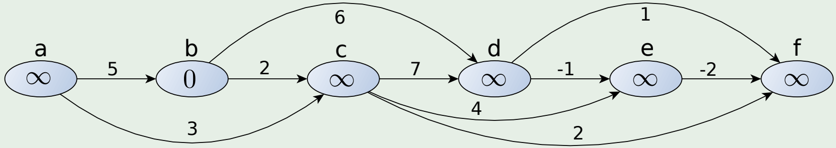
Outline

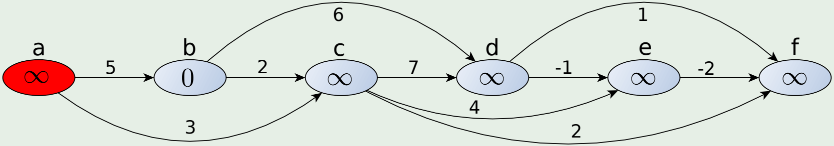
Example

After Initialization, we have **b** is the source



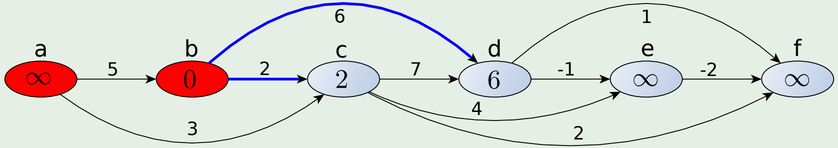
Example

a is the first in the topological sort, but no update is done



Example

b is the next one



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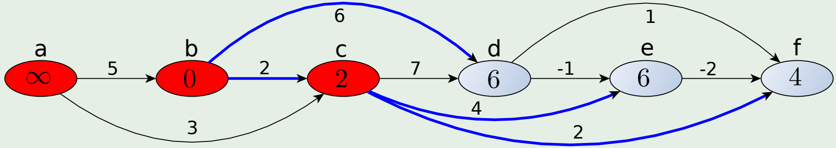
c is the next one



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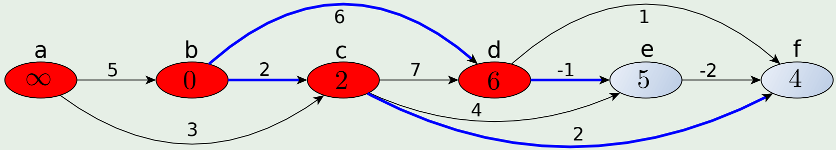

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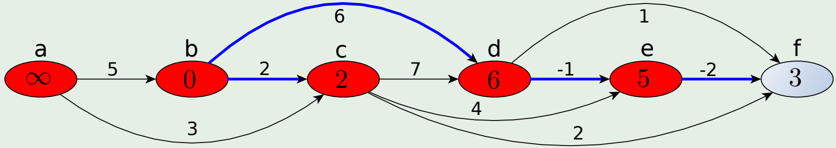
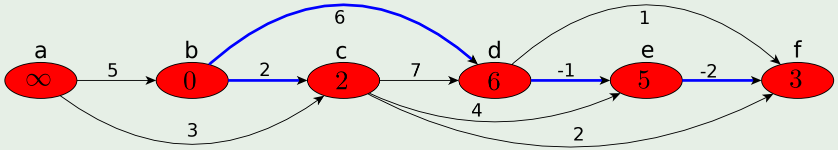


Example

Finally, **w** is the next one



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It is a greedy based method

Ideas?

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We need to keep track of the greedy choice!!!

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Assume no negative weight edges

1. **Dijkstra's algorithm maintains a set S of vertices whose shortest path from s has been determined.**
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DIJKSTRA($G, w, s$)
INITIALIZE($G, s$)
$S = \emptyset$
$Q = V[G]$
while $Q \neq \emptyset$
    $u$ =Extract-Min($Q$)
    $S = S \cup \{u\}$
    for each vertex $v \in Adj[u]$
        Relax(u,v,w)
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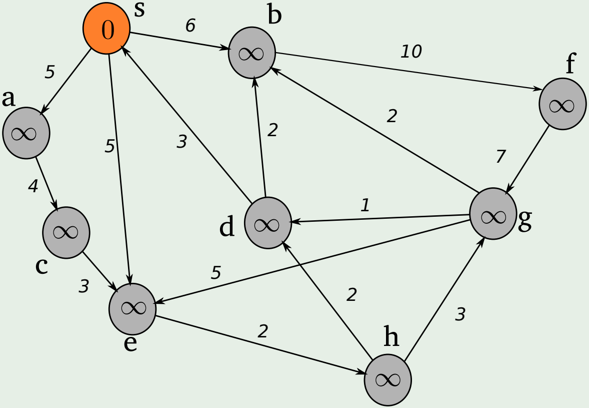
Outline

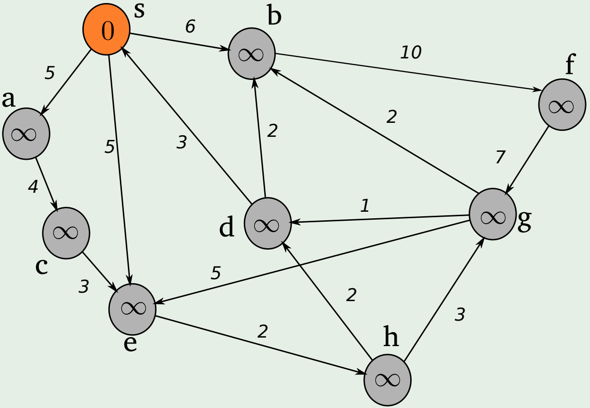
Example

The Graph After Initialization



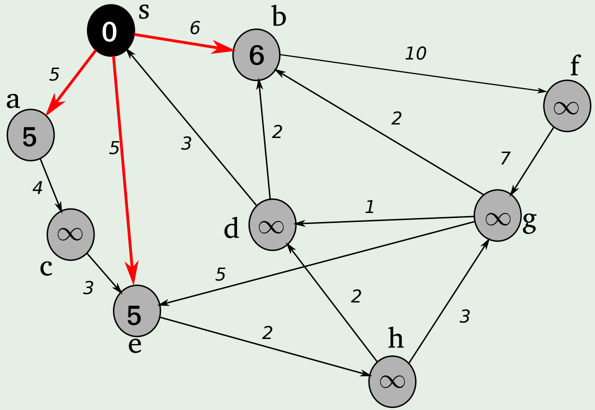
Example

We use red edges to represent $v.\pi$ and color black to represent the set S



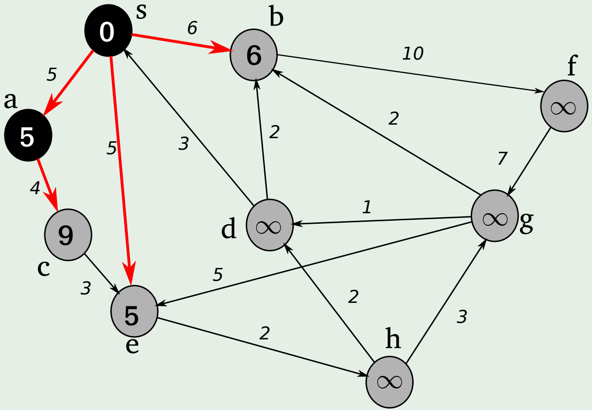
Example

$s \leftarrow$Extract-Min(Q) and update the elements adjacent to s



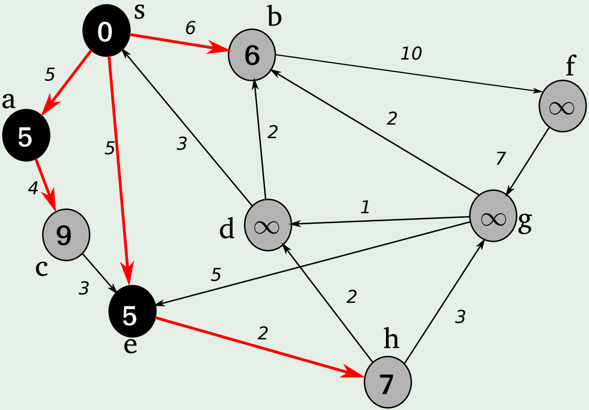
Example

a←Extract-Min(Q) and update the elements adjacent to a



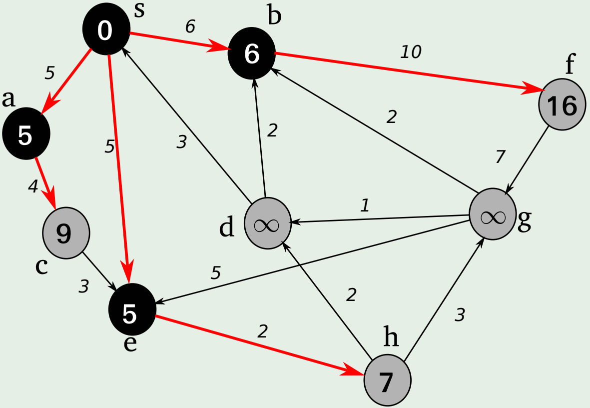
Example

e←Extract-Min(Q) and update the elements adjacent to e



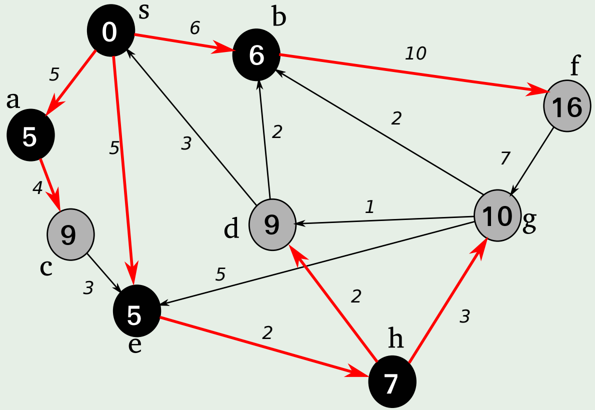
Example

b←Extract-Min(Q) and update the elements adjacent to b



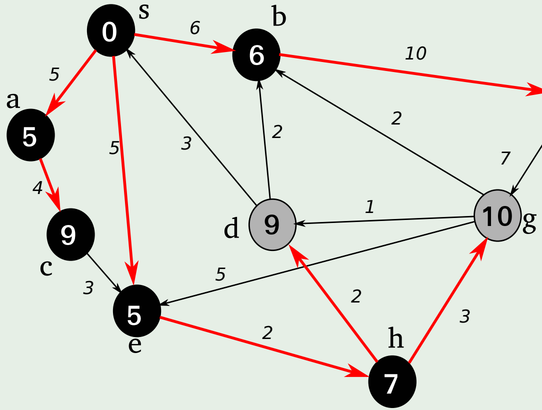
Example

h←Extract-Min(Q) and update the elements adjacent to h



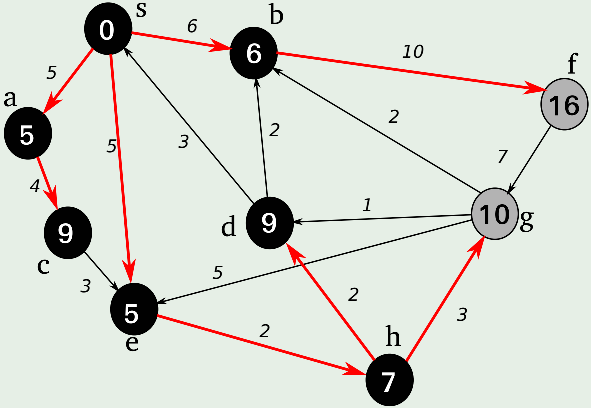
Example

c←Extract-Min(Q) and no-update



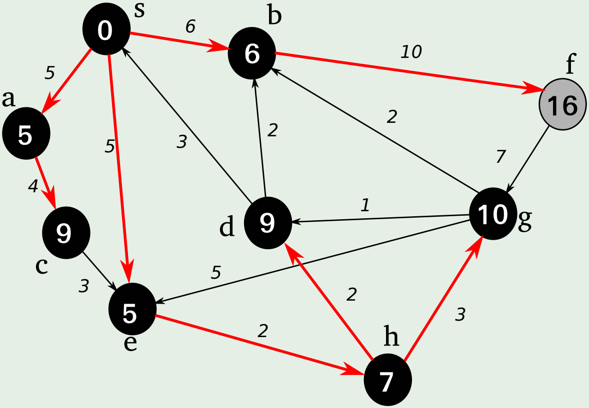
Example

d←Extract-Min(Q) and no-update



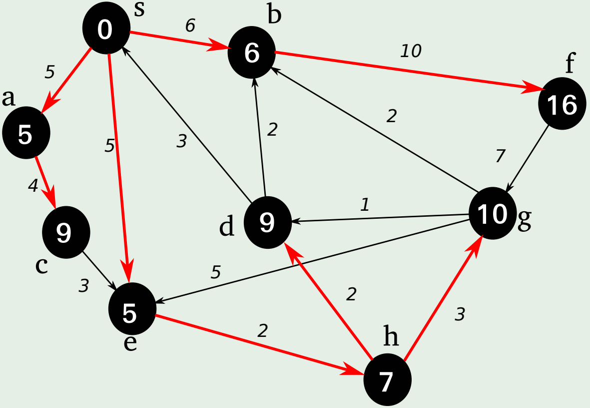
Example

g←Extract-Min(Q) and no-update



Example

f←Extract-Min(Q) and no-update



Outline

Correctness Dijkstra's algorithm

Theorem 24.6

Upon termination, $u.d = \delta(s, u)$ for all u in V (assuming non negative weights).

Proof

By lemma 24.11, once $u.d = \delta(s, u)$ holds, it continues to hold.

We are going to use the following loop Invariance

At the start of each iteration of the while loop of lines 4–8, $v.d = \delta(s, v)$ for each vertex $v \in S$.

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Thus

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Initially $S = \emptyset$, thus the invariant is true.

- In addition, we have that $S \neq \emptyset$ before u is added.

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We want to show that in each iteration $u.d = \delta\left(s, u\right)$ for the vertex added to set S.

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For this, note the following

- Note that $s.d = \delta(s, s) = 0$ when s is inserted, so $u \neq s$.
- In addition, we have that $S \neq \emptyset$ before u is added.

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Use contradiction

Now, suppose not. Let u be the first vertex such that $u.d \neq \delta$ (s,u) when inserted in S.

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Now

Note that there exist a path from s to u, for otherwise $u.d = \delta(s, u) = \infty$ by corollary 24.12.

- **“If there is no path from s to v, then $v.d = \delta(s, v) = \infty$ is an invariant.”**

Thus exist a shortest path p

Between s and u.

Observation

Prior to adding u to S, path p connects a vertex in S, namely s, to a vertex in $V - S$, namely u.

Proof

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Proof

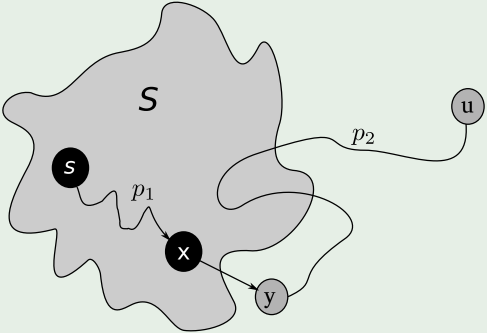
Consider the following

- The first y along p from s to u such that $y \in V - S$.
- And let $x \in S$ be y's predecessor along p.

Proof

Proof (continuation)

Then, shortest path from s to u: $s \overset{p_1}{\rightsquigarrow} x \to y \overset{p_2}{\rightsquigarrow} u$ looks like...



Remark: Either of paths p_1 or p_2 may have no edges.

Proof

We claim

$y.d = \delta(s, y)$ when u is added into S.

Proof of the claim

1. Observe that $x \in S$.

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Proof of the claim

1. Observe that $x \in S$.
2. In addition, we know that u is the first vertex for which $u.d \neq \delta(s, u)$ when it id added to S

Proof

Then

In addition, we had that $x.d = \delta(s, x)$ when x was inserted into S.

Proof

Then

In addition, we had that $x.d = \delta(s,x)$ when x was inserted into S.

Then, we relaxed the edge between x and y

Edge (x,y) was relaxed at that time!

Proof

Remember? Convergence property (Lemma 24.14)

Let p be a shortest path from s to v, where $p = s \overset{p_1}{\rightsquigarrow} u \rightarrow v$. If $u.d = \delta(s, u)$ holds at any time prior to calling $Relax(u, v, w)$, then $v.d = \delta(s, v)$ holds at all times after the call.

$$y.d = \delta(s, y) = \delta(s, x) + w(x, y) \tag{9}$$

- The claim is implied!!!

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Proof

Now

1. We obtain a contradiction to prove that $u.d = \delta\left(s,u\right)$.
2. y appears before u in a shortest path on a shortest path from s to u.
3. In addition, all edges have positive weights.
4. Then, $\delta(s,y) \leq \delta(s,u)$, thus

$$\begin{aligned} y.d &= \delta\left(s,y\right) \\ &\leq \delta\left(s,u\right) \\ &\leq u.d \end{aligned}$$

    - The last inequality is due to the Upper-Bound Property (Lemma 24.11).

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But because both vertices u and y where in $V - S$ when u was chosen in line 5 $\Rightarrow u.d \leq y.d$.

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- Conclusion: $u.d = \delta(s, u)$ when u is added to S and the equality is maintained afterwards

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- We have that $u.d = \delta\left(s, u\right)$, which contradicts our choice of u.
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Termination

- At termination $Q = \emptyset$
- Thus, $V - S = \emptyset$ or equivalent $S = V$

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$u.d = \delta(s, u)$ for all vertices $u \in V$!!!

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Outline

Complexity

Running time is

$O(V^2)$ using linear array for priority queue.
$O((V+E)\log V)$ using binary heap.
$O(V \log V + E)$ using Fibonacci heap.

Exercises

From Cormen's book solve

- 24.1-1
- 24.1-3
- 24.1-4
- 23.3-1
- 23.3-3
- 23.3-4
- 23.3-6
- 23.3-7
- 23.3-8
- 23.3-10