# Single-Source Shortest Paths

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## 1 Introduction

Here, we look at the problem of going from a source s to a possible multiple destinations. Most of the work is in the slides, but I wanted to point some stuff in the names of each of the Lemmas involved in proving the algorithms for finding the shortest paths.

## 2 Problem Definition

Formally the problem can be defined using the following concepts:

1. We have a weighted directed graph $G = (V, E)$
2. An associated weight function $w : E \rightarrow \mathbb{R}$ mapping edges to real-valued weights.
3. We define weight $w(p)$ of a path $p = \langle v_0, v_1, ..., v_k \rangle$ is defined as

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}v_i). \tag{1}$$

4. A shortest-path weight $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\left\{ w(p) \mid u \overset{p}{\rightsquigarrow} v \right\} & \text{if there is a path from } u \rightsquigarrow v \\ \infty & \text{otherwise} \end{cases}. \tag{2}$$

Thus a shortest path from vertex $u$ to $v$ can be defined as any path $p$ such that $w(p) = \delta(u, v)$.

## 3 Optimal substructure of a shortest path

All the following algorithms relay in the optimal substructure of the shortest path

1. Bellman-Ford Algorithm
2. DAG Algorithm
3. Dijkstra's Algorithm (Greedy Algorithm)

4. Edmond-Karp Algorithm
5. Floyd-Warshall Algorithm (Dynamic-programming Algorithm)

Thus, it is clear that we need to analyze and prove the lemmas in this chapter.

# 4 General Results

***Lemma 24.1 (Subpaths of shortest paths are shortest paths)***
Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from vertex $v_0$ to vertex $v_k$ and, for any $i$ and $j$ such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of $p$ from vertex $v_i$ to vertex $v_j$. Then, $p_{ij}$ is a shortest path from $v_i$ to $v_j$.

***Proof*** If we decompose path $p$ into $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. Now, assume that there is a path $p'_{ij}$ from $v_i$ to $v_j$ with weight $w(p'_{ij}) < w(p_{ij})$. Then, $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p'_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$ is a path from $v_0$ to $v_k$ whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that $p$ is a shortest path from $v_0$ to $v_k$. ■

***Lemma 24.10 (Triangle inequality)***
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$ and source vertex $s$. Then, for all edges $(u, v) \in E$, we have

$$\delta(s, v) \le \delta(s, u) + w(u, v) .$$

***Proof*** Suppose that $p$ is a shortest path from source $s$ to vertex $v$. Then $p$ has no more weight than any other path from $s$ to $v$. Specifically, path $p$ has no more weight than the particular path that takes a shortest path from source $s$ to vertex $u$ and then takes edge $(u, v)$.

Exercise 24.5-3 asks you to handle the case in which there is no shortest path from $s$ to $v$. ■

***Lemma 24.11 (Upper-bound property)***
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$. Let $s \in V$ be the source vertex, and let the graph be initialized by INITIALIZE-SINGLE-SOURCE$(G, s)$. Then, $v.d \geq \delta(s, v)$ for all $v \in V$, and this invariant is maintained over any sequence of relaxation steps on the edges of $G$. Moreover, once $v.d$ achieves its lower bound $\delta(s, v)$, it never changes.

***Proof*** We prove the invariant $v.d \geq \delta(s, v)$ for all vertices $v \in V$ by induction over the number of relaxation steps.

For the basis, $v.d \geq \delta(s, v)$ is certainly true after initialization, since $v.d = \infty$ implies $v.d \geq \delta(s, v)$ for all $v \in V - \{s\}$, and since $s.d = 0 \geq \delta(s, s)$ (note that $\delta(s, s) = -\infty$ if $s$ is on a negative-weight cycle and 0 otherwise).

For the inductive step, consider the relaxation of an edge $(u, v)$. By the inductive hypothesis, $x.d \geq \delta(s, x)$ for all $x \in V$ prior to the relaxation. The only $d$ value that may change is $v.d$. If it changes, we have

$$\begin{aligned} v.d &= u.d + w(u, v) \\ &\geq \delta(s, u) + w(u, v) \quad \text{(by the inductive hypothesis)} \\ &\geq \delta(s, v) \quad \text{(by the triangle inequality)}\ , \end{aligned}$$

and so the invariant is maintained.

To see that the value of $v.d$ never changes once $v.d = \delta(s, v)$, note that having achieved its lower bound, $v.d$ cannot decrease because we have just shown that $v.d \geq \delta(s, v)$, and it cannot increase because relaxation steps do not increase $d$ values. ■

***Lemma 24.13***
Let $G = (V, E)$ be a weighted, directed graph with weight function $w : E \to \mathbb{R}$, and let $(u, v) \in E$. Then, immediately after relaxing edge $(u, v)$ by executing RELAX$(u, v, w)$, we have $v.d \leq u.d + w(u, v)$.

***Proof*** If, just prior to relaxing edge $(u, v)$, we have $v.d > u.d + w(u, v)$, then $v.d = u.d + w(u, v)$ afterward. If, instead, $v.d \leq u.d + w(u, v)$ just before the relaxation, then neither $u.d$ nor $v.d$ changes, and so $v.d \leq u.d + w(u, v)$ afterward. ■

# 5 Lemma's Names

We have the following names for several of the lemmas in the slides, they tend to be quite enlightening:

- **Triangle inequality** (Lemma 24.10)
- **Upper-bound property** (Lemma 24.11)
- **No-path property** (Corollary 24.12)
- **Convergence property** (Lemma 24.14)
- **Path-relaxation property** (Lemma 24.15)
- **Predecessor-subgraph property** (Lemma 24.17)

# 6 Notes about some Proofs

## 6.1 Proof Lemma 24.16

*Proof.* Then, we have two simple paths from $s$ to $v$:

- $p_1$, which can be decomposed into $s \rightsquigarrow u \rightsquigarrow x \to z \rightsquigarrow v$.
- $p_2$, which can be decomposed into $s \rightsquigarrow u \rightsquigarrow y \to z \rightsquigarrow v$.

with $x \neq y$. However, $z.\pi = x$ and $z.\pi = y$ or $x = y$ a contradiction. Thus, $G_\pi$ contains a simple path from $s$ to $v$, thus forms a rooted tree $G_\pi$ with root $s$. □