Analysis of Algorithms

Fibonacci Heaps

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October 29, 2015

Outline

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Some previous definitions

Free Tree

A free tree is a connected acyclic undirected graph.

Rooted Tree

A rooted tree is a free tree in which one of the nodes is a root.

Ordered Tree

An ordered tree is a rooted tree where the children are ordered.

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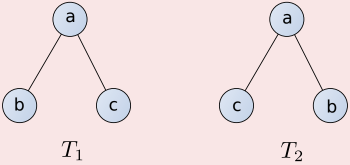

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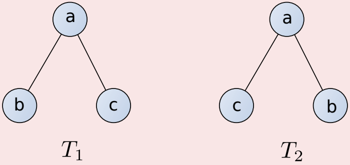

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Types of Ordered Trees

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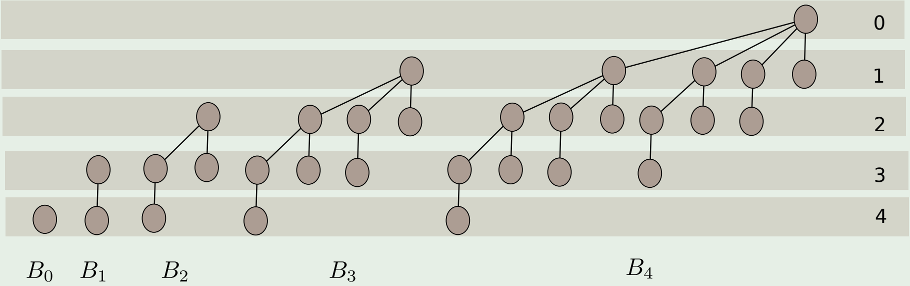
Examples

Recursive Structure



This can be seen too as

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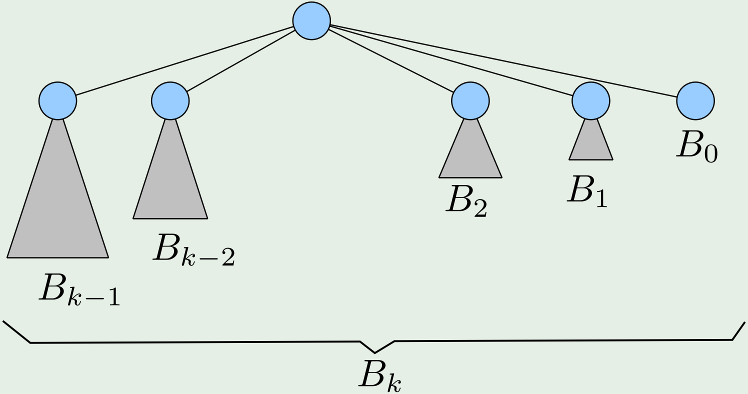

Properties of binomial trees

Lemma 19.1

For the binomial tree B_k:

1. There are 2^k nodes.
2. The height of the tree is k.
3. There are exactly $\binom{k}{i}$ nodes at depth i for $i = 0, 1, ..., k$.
4. The root has degree k, which is greater than that of any other node; moreover if the children of the root are numbered from left to right by $k-1, k-2, ..., 0$ child i is the root of a subtree B_i.

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Mergeable Heap Operations

The Fibonacci heap data structure is used to support the operations "meargeable heap" operations

1. MAKE -HEAP() creates and returns a new heap containing no elements.
2. INSERT(H, x) inserts element x, whose key has already been filled in, into heap H.
3. MINIMUM(H) returns a pointer to the element in heap H whose key is minimum.
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6. DECREASE-KEY(H, x, k) assigns to element x within heap H the new key value k.
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Each tree obeys the min-heap property:

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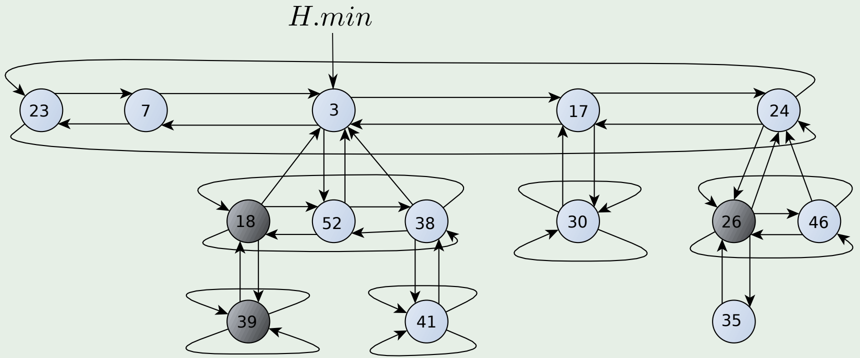

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Why Fibonacci Heaps?

Fibonacci Heaps facts

- Fibonacci heaps are especially desirable when the number of calls to Extract-Min and Delete is small.
- All other operations run in $O(1)$.

Applications

Fibonacci heaps may be used in many applications. Some graph problems, like minimum spanning tree and single-source-shortest-path.

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It is more...

We have that

Procedure	Binary Heap (Worst Case)	Fibonacci Heap (Amortized)
Make-Heap	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(\log n)$	$\Theta(1)$
Minimum	$\Theta(1)$	$\Theta(1)$
Extract-Min	$\Theta(\log n)$	$\Theta(\log n)$
Union	$\Theta(n)$	$\Theta(1)$
Decrease-Key	$\Theta(\log n)$	$\Theta(1)$
Delete	$\Theta(\log n)$	$\Theta(\log n)$

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Fields in each node

The Classic Ones

Each node contains a $x.parent$ and $x.child$ field.

The ones for the doubled linked list

The children of each a node x are linked together in a circular double linked list:

- Each child y of x has a $y.left$ and $y.right$ to do this.

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Thus, we have the following important labels

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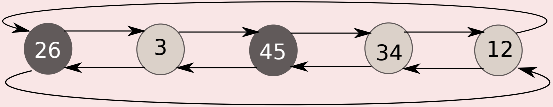

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Idea behind Fibonacci heaps

Main idea

Fibonacci heaps are called lazy data structures because they delay work as long as possible using the field $mark$!!!

Make Heap

Make a heap

You only need the following code:

MakeHeap()

1. $min[H] = NIL$
2. $n(H) = 0$

Complexity is as simple as

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Insertion

Code for Inserting a node

Fib-Heap-Insert(H, x)

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$x.degree = 0$
$x.p = NIL$
$x.child = NIL$
$x.mark = FALSE$
if $H.min = NIL$
    Create a root list for $H$ containing just $x$
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    if $x.key < H.min.key$
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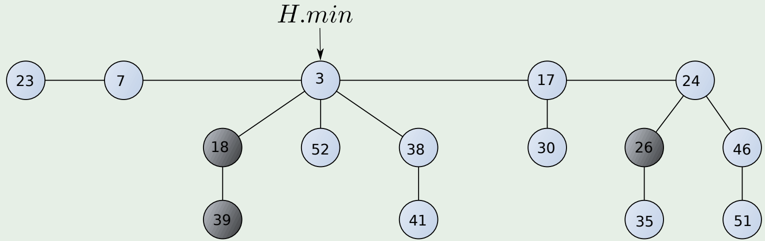

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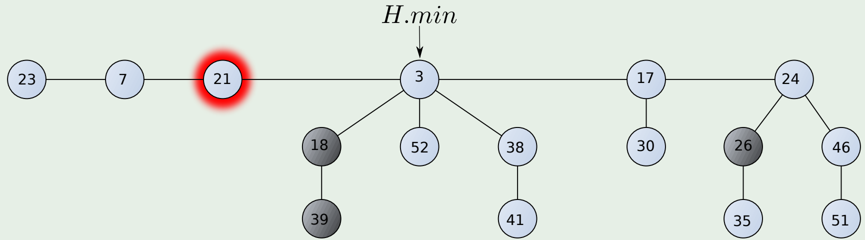

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Finding the Minimum

- Simply return the key of $min(H)$.
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What about Union of two Heaps?

Code for Union of Heaps

```
Fib-Heap-Union($H_1, H_2$)
    $H$ =Make-Fib-Heap()
    $H.min = H_1.min$
    Concatenate the root list of $H_2$ with the root list of $H$
    if ($H_1.min == NIL$) or ($H.min \neq NIL$ and $H_2.min.key < H_1.min.key$)
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    $H.n = H_1.n + H_2.n$
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In order to analyze Union...

We introduce some ideas from...

Our old friend amortized analysis and potential method!!!

Amortized potential function

We have the following function

$$\Phi(H) = t(H) + 2m(H)$$

- Where:
    - $t(H)$ is the number of trees in the Fibonacci heap
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About the known bound $D(n)$

- $D(n)$ is the maximum degree of any node in the binomial heap.
- It is more if the Fibonacci heap is a collection of unordered trees, then $D(n) = \log n$.
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Observations about $D(n)$

About the known bound $D(n)$

- $D(n)$ is the maximum degree of any node in the binomial heap.
- It is more if the Fibonacci heap is a collection of unordered trees, then $D(n) = \log n$.
    - We will prove this latter!!!

Back to Insertion

First

If H' is the Fibonacci heap after inserting, and H before that:

$$t(H') = t(H) + 1$$

Second

$$m(H') = m(H)$$

Then the change of potential is

$\Phi(H') - \Phi(H) = 1$ then complexity analysis results in $O(1) + 1 = O$

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Other operations: Find Min

It is possible to rephrase this in terms of potential cost

By using the pointer $min[H]$ potential cost is 0 then $O(1)$.

Other operations: Union

Union of two Fibonacci heaps

```
Fib-Heap-Union($H_1$, $H_2$)
    $H$ =Make-Fib-Heap()
    $H.min = H_1.min$
    Concatenate the root list of $H_2$ with the root list of $H$
    If ($H_1.min == NIL$) or ($H.min \neq NIL$ and $H_2.min.key < H_1.min.key$)
        $H.min = H_2.min$
    $H.n = H_1.n + H_2.n$
    return $H$
```

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Union of two Fibonacci heaps

Fib-Heap-Union(H_1, H_2)

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H = Make-Fib-Heap()
H.min = H_1.min
Concatenate the root list of H_2 with the root list of H
If (H_1.min == NIL) or (H.min ≠ NIL and H_2.min.key < H_1.min.key)
    H.min = H_2.min
H.n = H_1.n + H_2.n
return H
```

Cost of uniting two Fibonacci heaps

First

- $t(H) = t(H_1) + t(H_2)$ and $m(H) = m(H_1) + m(H_2)$.

Second

- $c_i = O(1)$ this is because the number of steps to make the union operations is a constant.

Potential analysis

$$\widehat{c}_i = c_i + \Phi(H) - [\Phi(H_1) + \Phi(H_2)] = O(1) + 0 = O(1).$$

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Extract min

Extract min

Fib-Heap-Extract-Min(H)

```
$z = H.min$
if $z \neq NIL$
    for each child $x$ of $z$
        add $x$ to the root list of $H$
        $x.p = NIL$
    remove $z$ from the root list of $H$
    if $z == z.right$
        $H.min = NIL$
    else $H.min = z.right$
        Consolidate$(H)$
    $H.n = H.n - 1$
return $z$
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What is happening here?

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Here, the code in lines 3-6 remove the node z and adds the children of z to the root list of H.

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Next

If the Fibonacci Heap is not empty a consolidation code is triggered.

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Thus

- The consolidate code is used to **eliminate** subtrees that have the same root degree by linking them.
- It repeatedly executes the following steps:
    1. Find two roots x and y in the root list with the same degree. Without loss of generality, let $x.key \leq y.key$.
    2. Link y to x: remove y from the root list, and make y a child of x by calling the FIB-HEAP-LINK procedure.
        - ★ This procedure increments the attribute $x.degree$ and clears the mark on y.

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Outline

Consolidate Code

Consolidate(H)

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Let $A(0 \ldots D(H.n))$ be a new array
for $i = 0$ to $D(H.n)$
    $A[i] = NIL$
for each $w$ in the root list of $H$
    $x = w$
    $d = x.degree$
    while $A[d] \neq NIL$
        $y = A[d]$
        if $x.key > y.key$
            exchange $x$ with $y$
        Fib-Heap-Link($H$, $y$, $x$)
        $A[d] = NIL$
        $d = d + 1$
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for $i = 0$ to $D(H.n)$
    if $A[i] \neq NIL$
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Fib-Heap-Link Code

Fib-Heap-Link(H, y, x)

1. Remove y from the root list of H
2. Make y a child of x, incrementing $x.degree$
3. $y.mark = FALSE$

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A while loop inside of the for loop - lines 1-15

- Using a w variable to go through the root list
- This is used to fill the pointers in $A\left[0...D\left(H.n\right)\right]$
- Then you link both trees using who has a larger key
- Then you add a pointer to the new min-heap, with new degree, in A.

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Lines 15-23 clean the original Fibonacci Heap, then using the pointers at the array A, each subtree is inserted into the root list of H.

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- This is used to fill the pointers in $A\left[0 \ldots D\left(H.n\right)\right]$
- Then you link both trees using who has a larger key
- Then you add a pointer to the new min-heap, with new degree, in A.

Then - lines 15-23

Lines 15-23 clean the original Fibonacci Heap, then using the pointers at the array A, each subtree is inserted into the root list of H.

Code Process

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We have the following Loop Invariance

At the start of each iteration of the while loop, $d = x.degree$.

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Line 6 ensures that the loop invariant holds the first time we enter the loop.

Maintenance

We have two nodes x and y such that they have the same degree then

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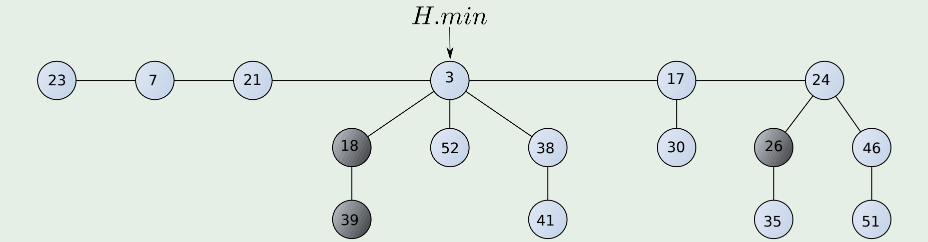
Loop Invariance

Termination

We repeat the while loop until $A\left[d\right] = NIL$, in which case there is no other root with the same degree as x.

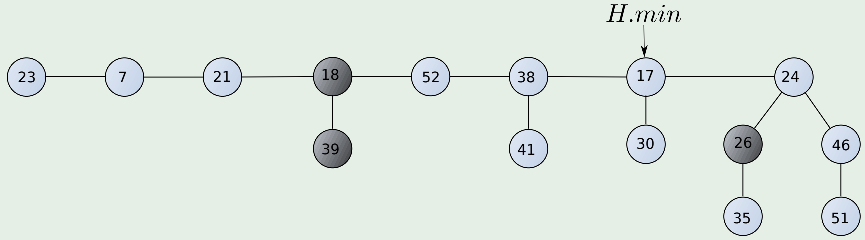
Example of consolidation

We remove $H.min == 3$



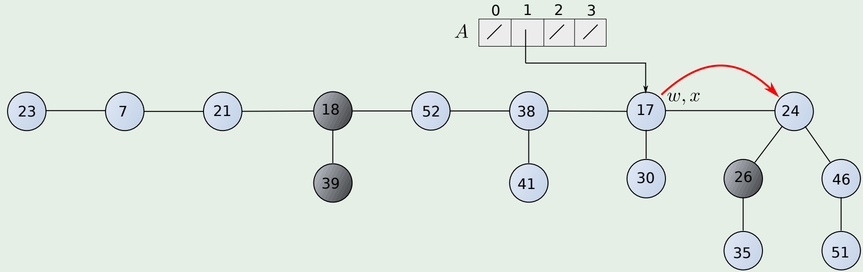
Example of consolidation

The children are moved to the root's list



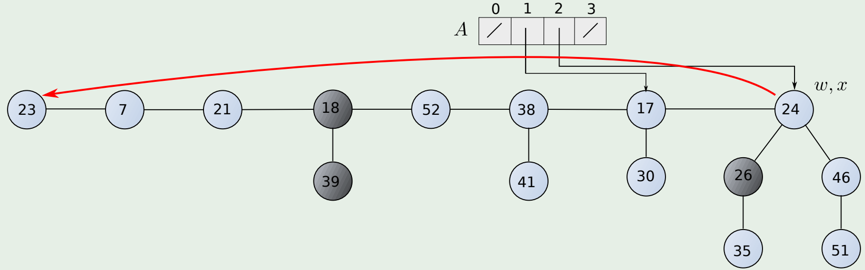
Example of consolidation

Now, you get Consolidation running beginning $A[1] \rightarrow 17$



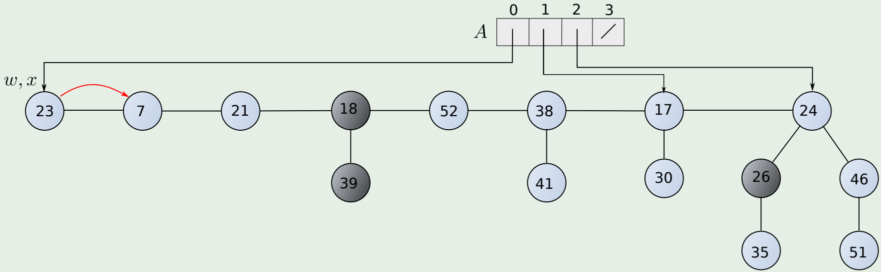
Example of consolidation

Now, $A[2] \rightarrow 24$



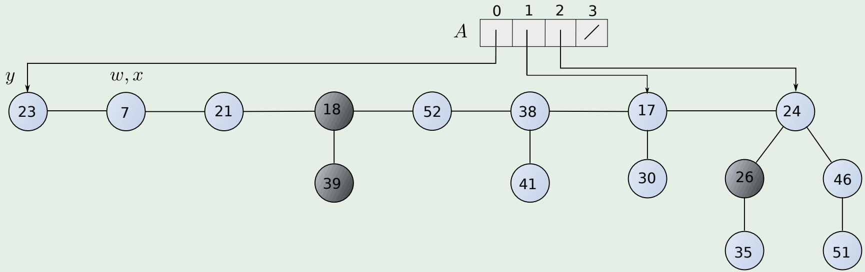
Example of consolidation

We have a pointer to a node with $degree = 0$



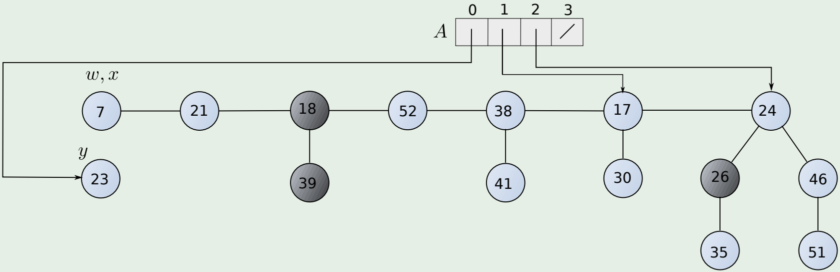
Example of consolidation

We don't do an exchange



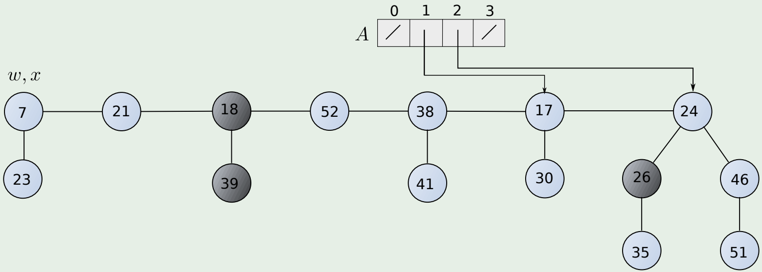
Example of consolidation

Remove y from the root's list



Example of consolidation

Make y a child of x



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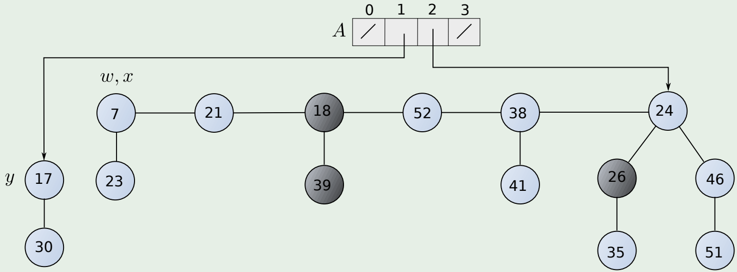

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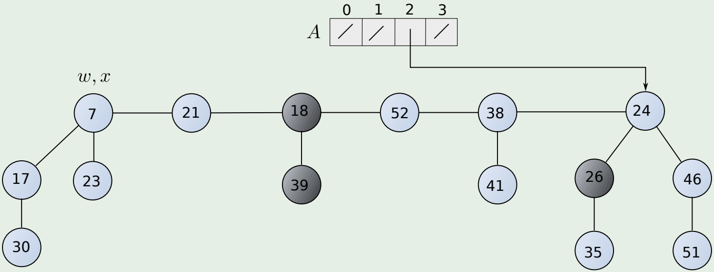
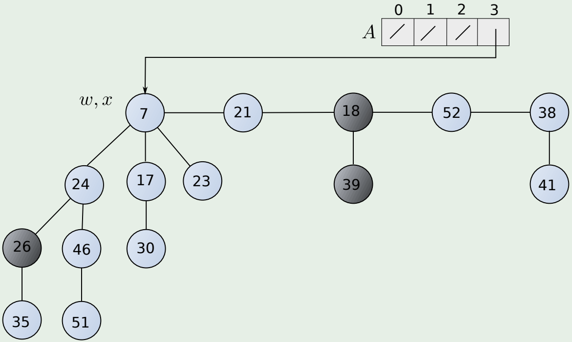


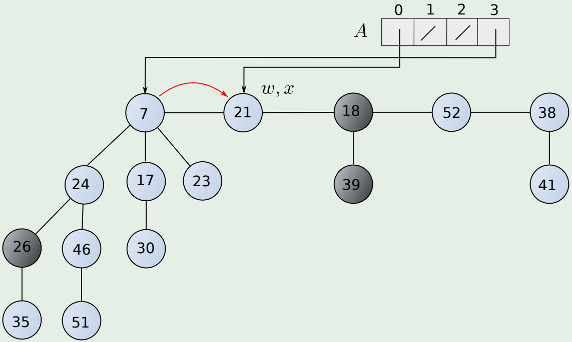
Example of consolidation

and we point to the the element with $degree = 3$



Example of consolidation

We move to the next root's child and point to it from A



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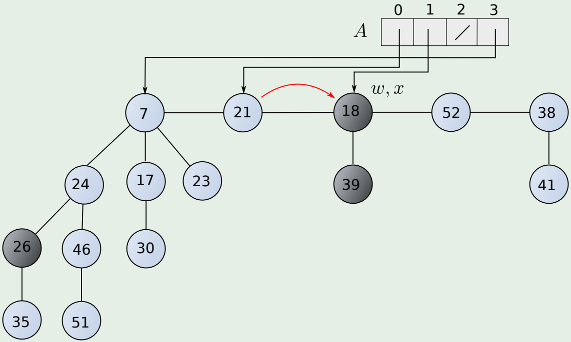

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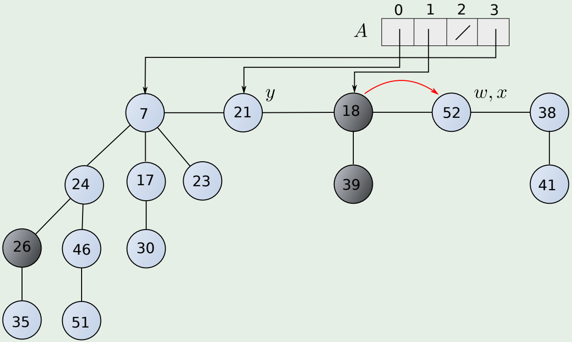
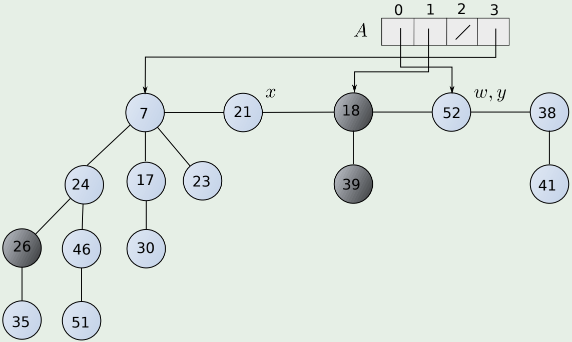


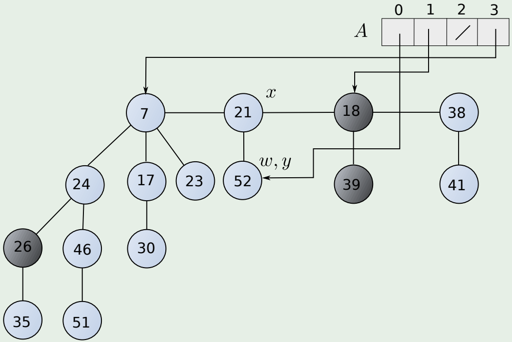
Example of consolidation

We point $y = A[0]$ then do the exchange between $x \longleftrightarrow y$



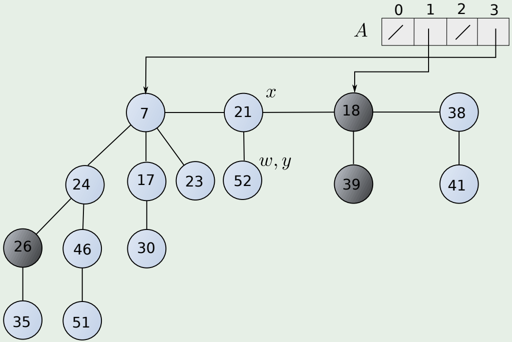
Example of consolidation

Link x and y



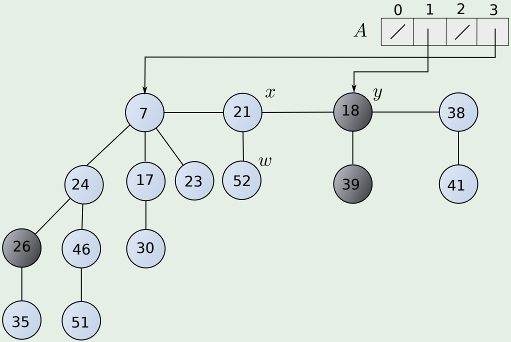
Example of consolidation

Make $A[d] = NIL$



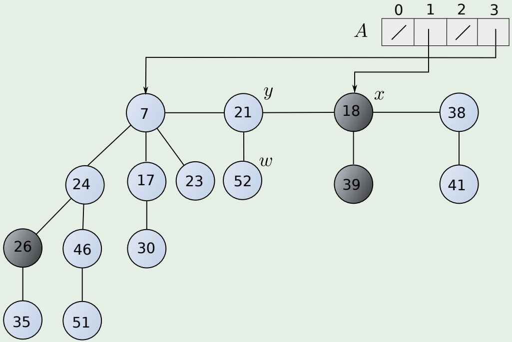
Example of consolidation

We make $y = A[1]$



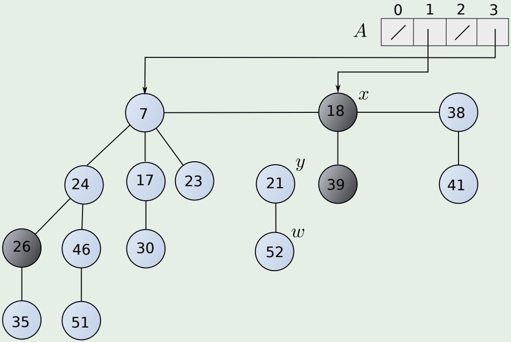
Example of consolidation

Do an exchange between $x \longleftrightarrow y$



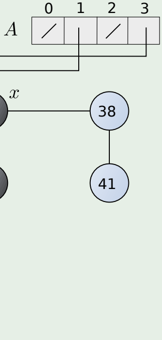
Example of consolidation

Remove y from the root's list



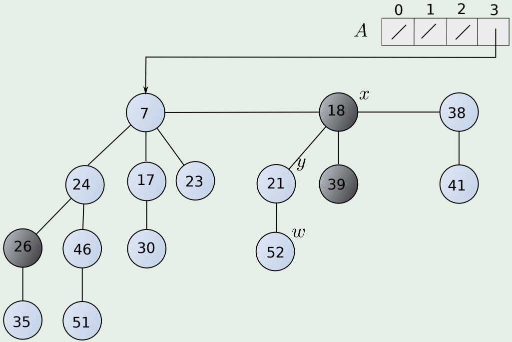
Example of consolidation

Make y a child of x



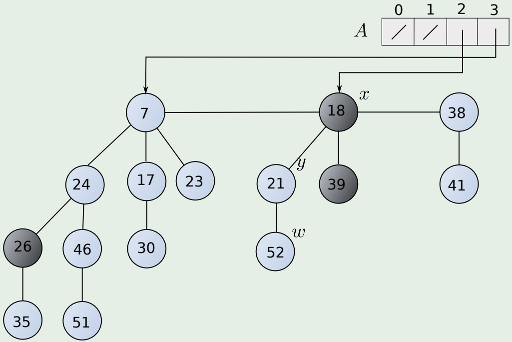
Example of consolidation

Make $A[1] = NIL$, then we make $d = d + 1$



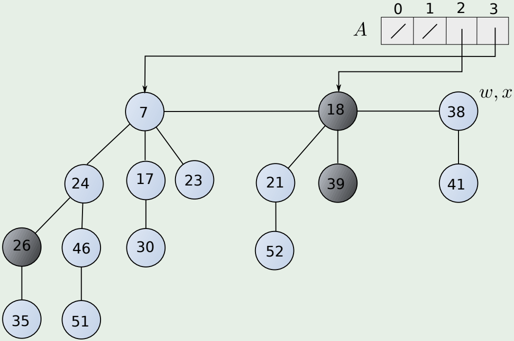
Example of consolidation

Because $A[2] = NIL$, then $A[2] = x$



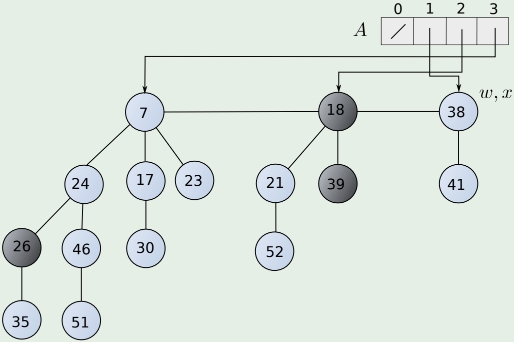
Example of consolidation

We move to the next w and make $x = w$



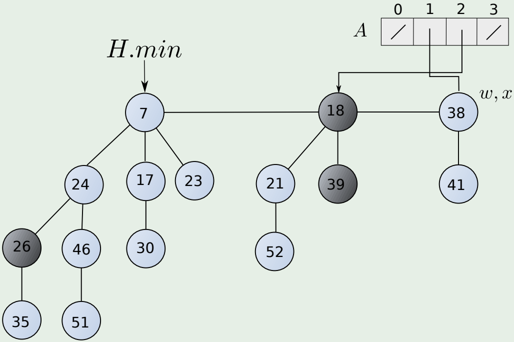
Example of consolidation

Because $A[1] = NIL$, then jump over the while loop and make $A[1] = x$



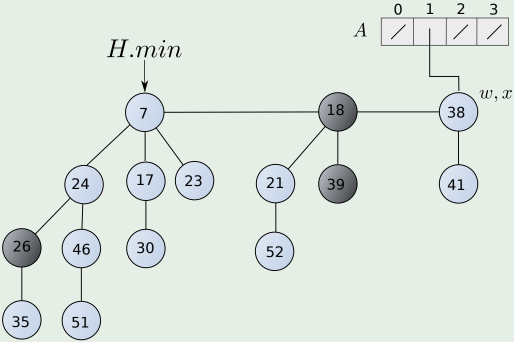
Example of consolidation

Because $A[1] \neq NIL$ insert into the root's list and because $H.min = NIL$, it is the first node in it



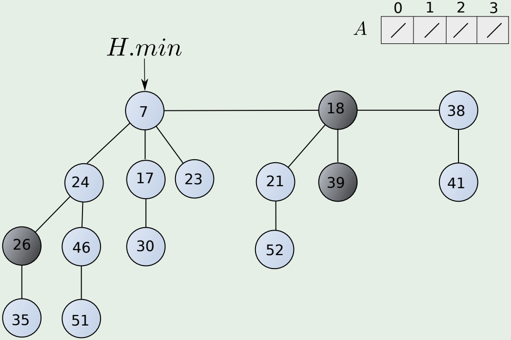
Example of consolidation

Because $A\left[2\right] \neq NIL$ insert into the root's list no exchange of min because $A\left[2\right].key > H.min.key$



Example of consolidation

Because $A[3] \neq NIL$ insert into the root's list no exchange of min because $A[3].key > H.min.key$



Outline

Cost of Extract-min

Amortized analysis observations

The cost of FIB-EXTRACT-MIN contributes at most $O\left(D\left(n\right)\right)$ because

1. The for loop at lines 3 to 5 in the code FIB-EXTRACT-MIN.
2. for loop at lines 2-3 and 16-23 of CONSOLIDATE.

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We have that

The size of the root list when calling Consolidate is at most

$$D(n) + t(H) - 1 \tag{1}$$

because the min root was extracted an it has at most $D(n)$ children.

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At lines 4 to 14 in the CONSOLIDATE code:

- The amount of work done by the **for** and the **while** loop is proportional to $D(n) + t(H)$ because each time we go through an element in the root list (for loop).
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Potential cost

Thus, assuming that H' is the new heap and H is the old one

- $\Phi\left(H\right) = t\left(H\right) + 2 \cdot m\left(H\right)$.
- $\Phi\left(H'\right) = D\left(n\right) + 1 + 2 \cdot m\left(H\right)$ because
  - H' has at most $D(n) + 1$ elements after consolidation
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Potential cost

The Final Potential Cost is

$$\begin{aligned}\widehat{c}_i &= c_i + \Phi\left(H'\right) - \Phi\left(H\right)\\ &= O\left(D\left(n\right) + t\left(H\right)\right) + D\left(n\right) + 1 + 2\cdot m\left(H\right) - t\left(H\right) - 2\cdot m\left(H\right)\\ &= O\left(D\left(n\right) + t\left(H\right)\right) - t\left(H\right)\\ &= O\left(D\left(n\right)\right).\end{aligned}$$

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$D(n) = O(\log n)$

Outline

Decreasing a key

Fib-Heap-Decrease-Key(H, x, k)

```
if $k > x.key$
        error "new key is greater than current key"
$x.key = k$
$y = x.p$
if $y \neq NIL$ and $x.key < y.key$
        CUT$(H, x, y)$
        Cascading-Cut$(H, y)$
if $x.key < H.min.key$
        $H.min = x$
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Explanation

First

1. Lines 1–3 ensure that the new key is no greater than the current key of x and then assign the new key to x.
2. If x is not a root and if $x.key \leq y.key$, where y is x's parent, then CUT and CASCADING-CUT are triggered.

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Decreasing a key (continuation - cascade cutting)

Be lazy to remove keys!

```
Cut($H, x, y$)
    Remove $x$ from the child list of $y$, decreasing $y.degree$
    Add $x$ to the root list of $H$
    $x.p = NIL$
    $x.mark = FALSE$
```

```
Cascading-Cut($H, y$)
    $z = y.p$
    if $z \neq NIL$
        if y.mark == FALSE
            y.mark=TRUE
        else
            Cut($H, y, z$)
            Cascading-Cut($H, y$)($H, z$)
```

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  4. $x.mark = FALSE$
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Cascading-Cut($H, y$)
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  2. if $z \neq NIL$
  3.     if y.mark == FALSE
  4.         y.mark=TRUE
  5.     else
  6.         Cut($H, y, z$)
  7.         Cascading-Cut($H, y$)($H, z$)
```

Explanation

Second

Then CUT simply removes x from the child-list of y.

Thus

The CASCADING-CUT uses the mark attributes to obtain the desired time bounds.

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Explanation

The mark label records the following events that happened to y:

1. At some time, y was converted into an element of the root list.
2. Then, y was linked to (made the child of) another node.
3. Then, two children of y were removed by cuts.

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As soon as the second child has been lost, we cut y from its parent, making it a new root.

- The attribute $y.mark$ is TRUE if steps 1 and 2 have occurred and one child of y has been cut.
- The CUT procedure, therefore, clears $y.mark$ in line 4, since it performs step 1.
- We can now see why line 3 of FIB-HEAP-LINK clears $y.mark$.

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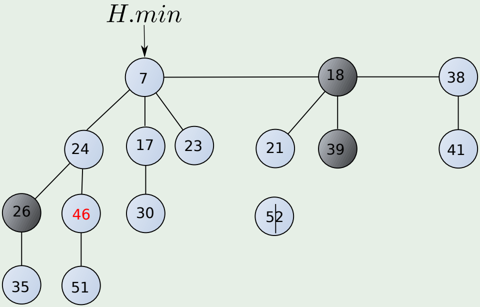
Example

46 is decreased to 15



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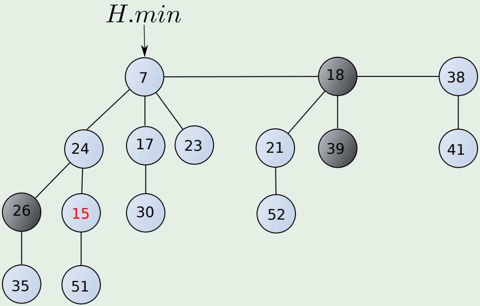
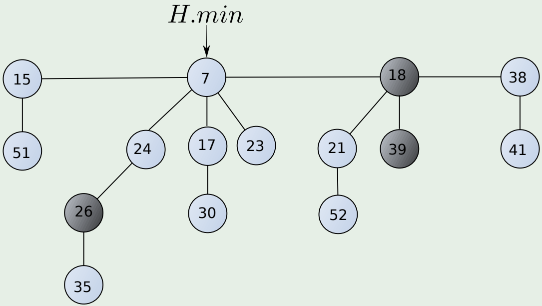
46 is decreased to 15



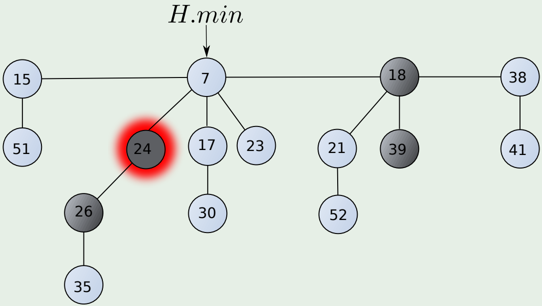
Example

Cut 15 to the root's list



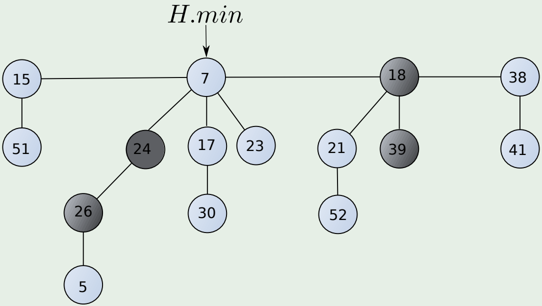
Example

Mark 24 to *True*



Example

Then 35 is decreased to 5



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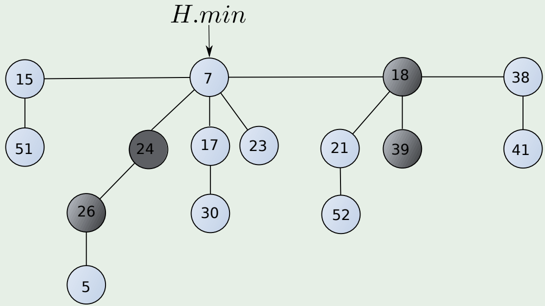
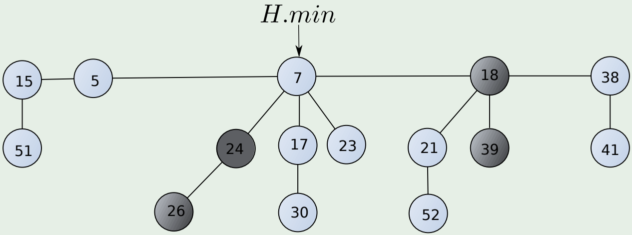


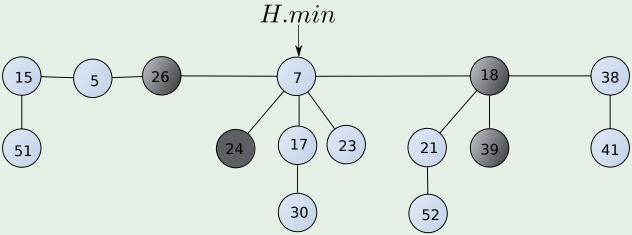
Example

Initiate cascade cutting



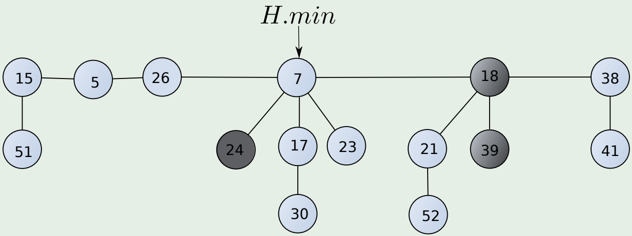
Example

Initiate cascade cutting moving 26 to the root's list



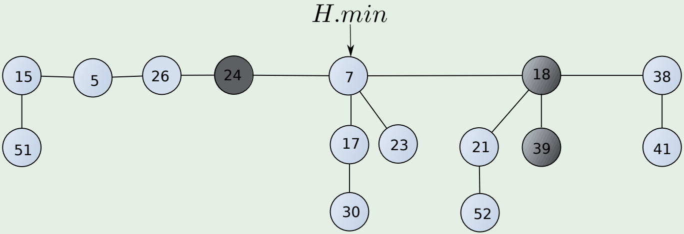
Example

Mark 26 to false



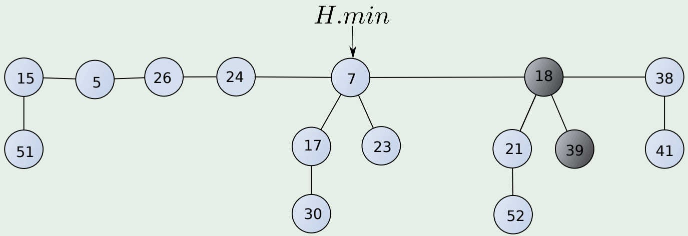
Example

Move 24 to root's list



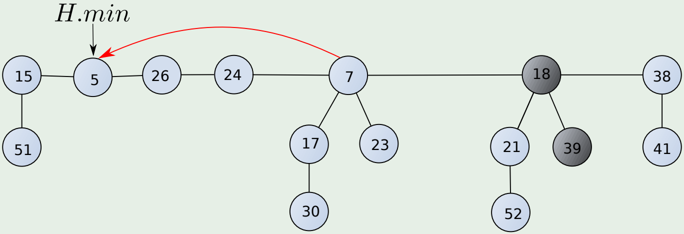
Example

Mark 26 to false



Example

Change the $H.min$



Potential cost

The procedure Decrease-Key takes

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Finally, assuming H' is the new Fibonacci Heap and H the old one

$$\Phi\left(H'\right) = \left(t\left(H\right) + c\right) + 2\left(m\left(H\right) - \left(c-1\right) + 1\right) \tag{2}$$

Where:

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Delete a node

It is easy to delete a node in the Fibonacci heap following the next code

```
Fib-Heap-Delete($H, x$)
  1 Fib-Heap-Decrease-Key($H, x, -\infty$)
  2 Fib-Heap-Extract-Min($H$)
```

Again, the cost is $O(D(n))$

Proving the $D(n)$ bound!!!

Let's define the following

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- $x.degree$ is always maintained as an accurate count of the degree of

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Let x be any node in a Fibonacci heap, and suppose that $x.degree = k$. Let $y_1, y_2, ..., y_k$ denote the children of x in the order in which they were linked to x, from the earliest to the latest. Then $y_1.degree \geq 0$ and $y_i.degree \geq i - 2$ for $i = 2, 3, ..., k$.

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Outline

1. Introduction
   - Basic Definitions
   - Ordered Trees
2. Binomial Trees
   - Example
3. Fibonacci Heap
   - Operations
   - Fibonacci Heap
   - Why Fibonacci Heaps?
   - Node Structure
   - Fibonacci Heaps Operations
     - Mergeable-Heaps operations - Make Heap
     - Mergeable-Heaps operations - Insertion
     - Mergeable-Heaps operations - Minimum
     - Mergeable-Heaps operations - Union
   - Complexity Analysis
   - Consolidate Algorithm
   - Potential cost
   - Operation: Decreasing a Key
   - Why Fibonacci?
4. Exercises
   - Some Exercises that you can try

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The kth Fibonacci number is defined by the recurrence

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For al integers $k \geq 0$

$$F_{k+2} = 1 + \sum_{i=0}^{k} F_i$$

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Proof by induction $k = 0$....

The use of the Golden Ratio

Lemma 19.3

Let x be any node in a Fibonacci heap, and let $k = x.degree$. Then, $F_{k+2} \geq \Phi^k$, where $\Phi = \frac{1+\sqrt{5}}{2}$ (The golden ratio).

Golden Ratio

Building the Golden Ratio

1. Construct a unit square.
2. Draw a line from the midpoint of one side to an opposite corner.
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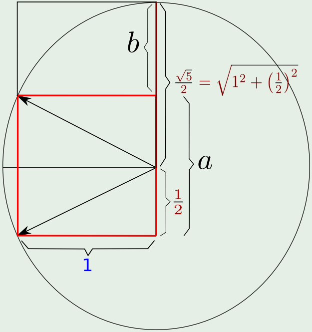
$$\frac{a+b}{a} = \frac{a}{b} = \varphi \quad a > b$$



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Let x be any node in a Fibonacci Heap, $k = x.degree$. Then $size(x) \geq F_{k+2} \geq \Phi^k$.

- Restrictions for s_k:
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As in the previous lemma

- $y_1, y_2, \ldots, y_k$ denote the children of x in the order they were linked to x.

The use of the Golden Ratio

Lemma 19.4

Let x be any node in a Fibonacci Heap, $k = x.degree$. Then $size(x) \geq F_{k+2} \geq \Phi^k$.

Proof

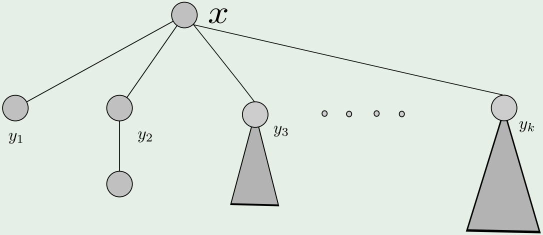
- Let s_k denote the minimum value for $size(x)$ over all nodes x such that $x.degree = k$.
- Restrictions for s_k:
    - $s_k \leq size(x)$ and $s_k \leq s_{k+1}$ (monotonically increasing).

As in the previous lemma

- $y_1, y_2, ..., y_k$ denote the children of x in the order they were linked to x.

Example

For example



Now

Look at the board

Example

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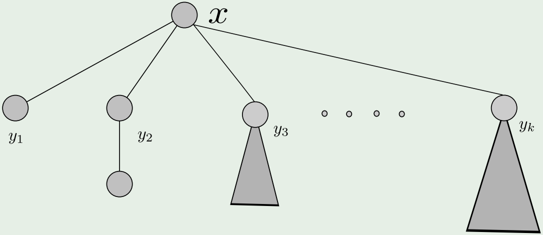

Now

Look at the board

The use of the Golden Ratio

Proof continuation

- Now, we need to proof that $s_k \geq F_{k+2}$
- The cases for $k = 0 \rightarrow s_0 = 1$, and $k = 1 \rightarrow s_1 = 2$ are trivial because

$$F_2 = F_1 + F_0 = 1 + 0 = 1 \tag{4}$$

$$F_3 = F_2 + F_1 = 1 + 1 = 2 \tag{5}$$

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Finally

Corollary 19.5

The maximum degree $D(n)$ of any node in an n-node Fibonacci heap is $O(\log n)$.

Proof

Look at the board!

Finally

Corollary 19.5

The maximum degree $D(n)$ of any node in an n-node Fibonacci heap is $O(\log n)$.

Proof

Look at the board!

Outline

Exercises

From Cormen's book solve the following

- 20.2-2
- 20.2-3
- 20.2-4
- 20.3-1
- 20.3-2
- 20.4-1
- 20.4-2