Analysis of Algorithms

Skip Lists

Andres Mendez-Vazquez

November 6, 2020

Outline

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Dictionaries

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A dictionary is a collection of elements; each of which has a unique search key.

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- Symbol table (with duplicates)
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Example: Course records

Dictionary with member records

key ID	Student Name	HW1	
123	Stan Smith	49	⋯
125	Sue Margolin	45	⋯
128	Billie King	24	⋯
⋮			
⋮			
190	Roy Miller	36	⋯

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The dictionary ADT operations

Some operations on dictionaries

- size(): Returns the size of the dictionary.
- empty(): Returns TRUE if the dictionary is empty.
- findItem(key): Locates the item with the specified key.
- findAllItems(key): Locates all items with the specified key.
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Example of unordered dictionary

Example

Consider an empty unordered dictionary, we have then...

Operation	Dictionary	Output
InsertItem$(5, A)$	$\{(5, A)\}$	
InsertItem$(7, B)$	$\{(5, A), (7, B)\}$	
findItem(7)	$\{(5, A), (7, B)\}$	B
findItem(4)	$\{(5, A), (7, B)\}$	No Such Key
size$()$	$\{(5, A), (7, B)\}$	2
removeItem(5)	$\{(7, B)\}$	A
findItem(4)	$\{(7, B)\}$	No Such Key

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How to implement a dictionary?

There are many ways of implementing a dictionary

- Sequences / Arrays
  - Ordered
  - Unordered
- Binary search trees
- Skip lists
- Hash tables

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Unordered array

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Binary searches

Features

- Narrow down the search range in stages
- "High-low" game.

Binary searches

Example find Element(22)

2	4	5	7	8	9	12	**14**	17	19	22	25	27	28	33
↑ LOW							↑ MID							↑ $HIGH$

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Recall binary search trees

Implement a dictionary with a BST

A binary search tree is a binary tree T such that:

- Each internal node stores an item (k, e) of a dictionary.
- Keys stored at nodes in the left subtree of v are less than or equal to k.
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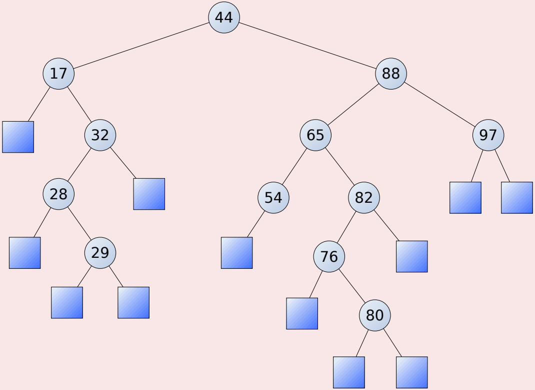
Binary searches Trees

Problem!!! Keeping a Well Balanced Binary Search Tree can be difficult!!!



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We want to have a

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Thus, we have the following possibilities

Unordered array complexities

Insertion: $O(1)$

Search: $O(n)$

Ordered array complexities

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Big Drawback - Complex parallel Implementation and waste of memory.

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We want something better!!!

For this

We will present a probabilistic data structure known as Skip List!!!

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Starting from Scratch

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- Imagine that you only require to have searches.
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- Then, using this How do we speed up searches?

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- The Top is the high way system, L_1.

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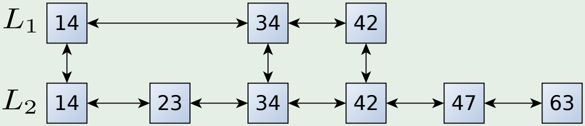
High-Bottom Way System



Thus, we have...

The following rule

To Search first search in the top one (L_1) as far as possible, then go down and search in the bottom one (L_2).

We can use a little bit of optimization

We have the following worst cost

$$\text{Search Cost High-Bottom Way System} = \text{Cost Searching Top} + \ldots$$

$$\text{Cost Search Bottom}$$

Or

$$\text{Search Cost} = length\left(L_1\right) + \text{Cost Search Bottom}$$

The interesting part is "Cost Search Bottom"

This can be calculated by the following quotient:

$$\frac{length\left(L_2\right)}{length\left(L_1\right)}$$

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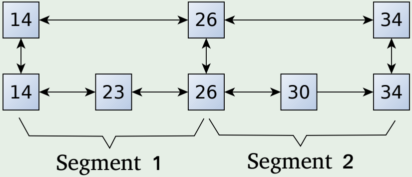
Why?

If we think we are jumping



Then cost of searching each of the bottom segments = 2

Thus the ratio is a "decent" approximation to the worst case search

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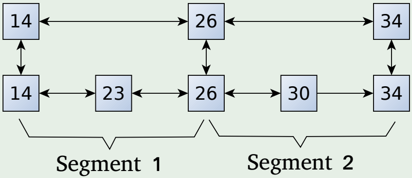

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Then, the cost for a search (when $length\left(L_2\right) = n$)

$$\text{Search Cost } = length\left(L_1\right) + \frac{length\left(L_2\right)}{length\left(L_1\right)} = length\left(L_1\right) + \frac{n}{length\left(L_1\right)} \quad (1)$$

Taking the derivative with respect to $length\left(L_1\right)$ and making the result equal 0

$$\frac{d\text{Search Cost}}{dlength\left(L_1\right)} = 1 - \frac{n}{length^2\left(L_1\right)} = 0$$

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Final Cost

We have that the optimal length for L_1

$$length\,(L_1) = \sqrt{n}$$

Plugging back in (Eq. 1)

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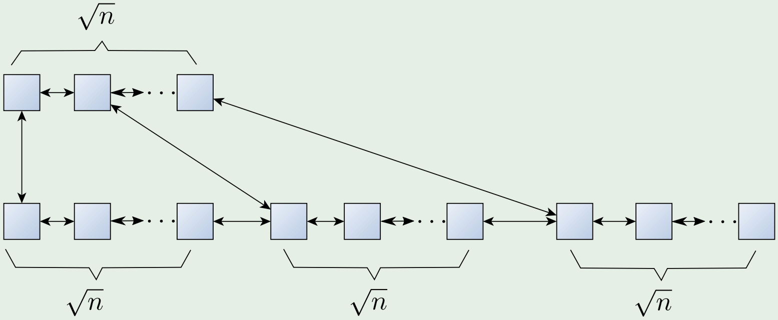
Data structure with a Square Root Relation

Something like this



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For a three layer link list data structure

We get a search cost of $3 \times \sqrt[3]{n}$

In general for k layers, we have

$$k \times \sqrt[k]{n}$$

Thus, if we make $k = \log_2 n$, we get

$$
\begin{aligned}
\text{Search Cost} &= \log_2 n \times \sqrt[\log_2 n]{n} \\
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&= \log_2 n \times 2 \\
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Something Notable

We get the advantages of the binary search trees with a simpler architecture!!!

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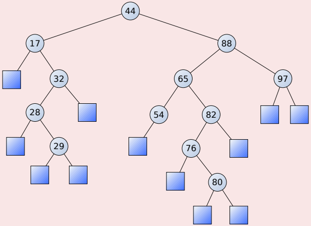
Binary Search Trees



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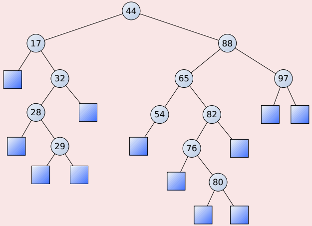

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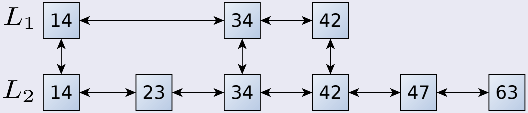

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Definition for Skip List

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Skip List Definition

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A skip list for a set S of distinct (key,element) items is a series of lists $S_0, S_1, ..., S_h$ such that:

- Each list S_i contains the special keys $+\infty$ and $-\infty$
- List S_0 contains the keys of S in nondecreasing order
- Each list is a subsequence of the previous one
  - $S_0 \supseteq S_1 \supseteq S_2 \supseteq ... \supseteq S_h$
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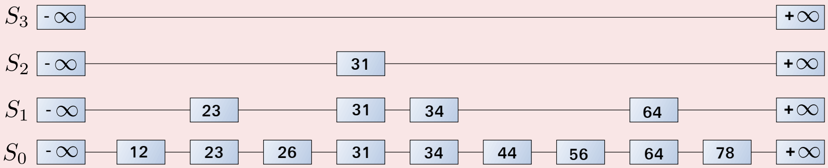

Skip list search

We search for a key x in a skip list as follows

- We start at the first position of the top list.
- At the current position p, we compare x with $y == p.next.key$
  - $x == y$: we return $p.next.element$
  - $x > y$: we scan forward
  - $x < y$: we "drop down"
- If we try to drop down past the bottom list, we return $null$.

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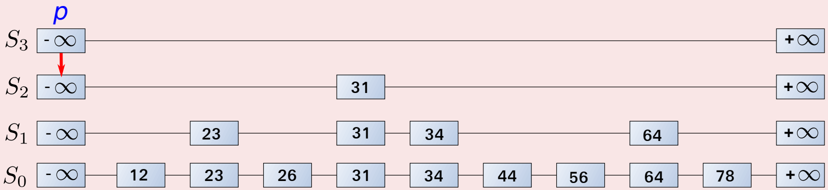
Example search for 78

$x < p.next.key$: "drop down"



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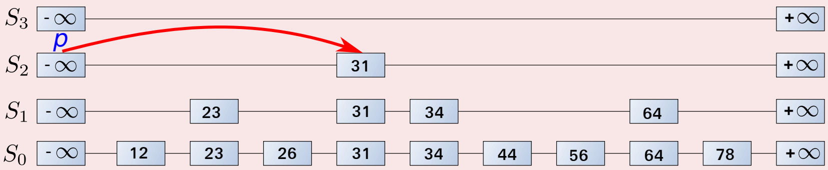

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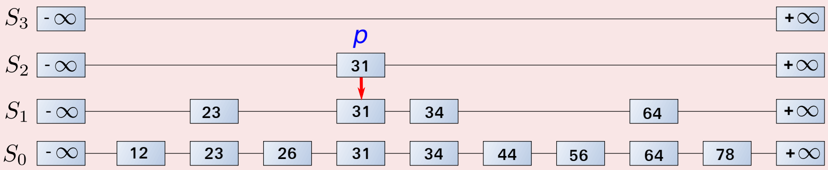

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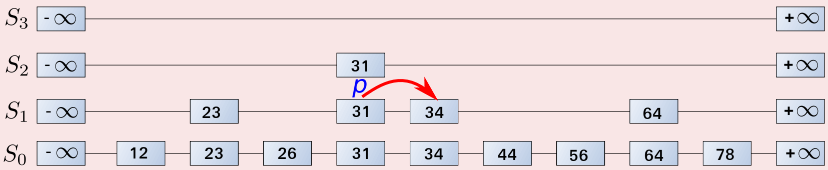

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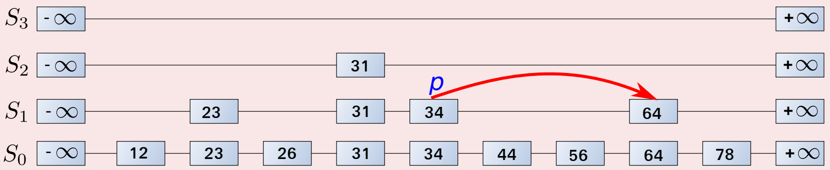

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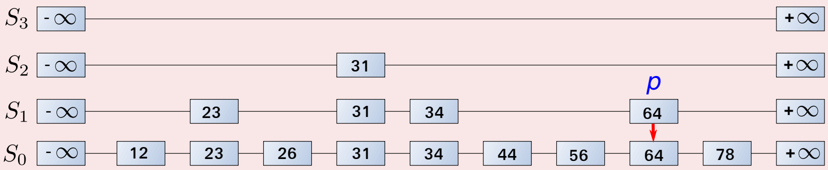

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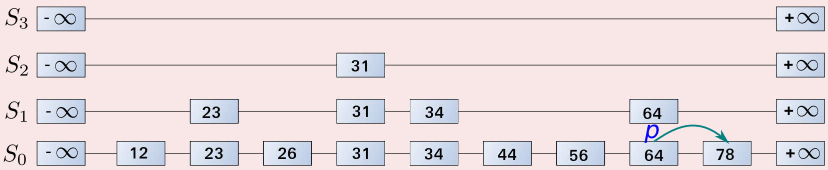
$x == y$: we return $p.next.element$



Outline

How do we implement this data structure?

We can implement a skip list with quad-nodes

A quad-node stores:

- Entry Value
- Link to the previous node
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Also we define special keys PLUS_INF and MINUS_INF, and we modify the key comparator to handle them.

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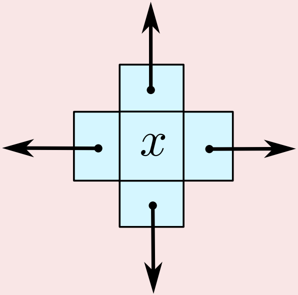
Quad-Node Example



Skip lists uses Randomization

Use of randomization

We use a randomized algorithm to insert items into a skip list.

Running time

We analyze the expected running time of a randomized algorithm under the following assumptions:

- The coins are unbiased.
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Outline

Insertion

To insert

To insert an entry $(key, object)$ into a skip list, we use a randomized algorithm:

1. We repeatedly toss a coin until we get tails.
   - We denote with i the number of times the coin came up heads.

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We have two cases

If $i \geq h$, we add to the skip list new lists $S_{h+1}, ..., S_{i+1}$

- Each containing only the two special keys.
- We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with largest key less than x in each lists $S_0, S_1, ..., S_i$.
- For $j \leftarrow 0, ..., i$, we insert item $(key, object)$ into list S_j after position p_j.

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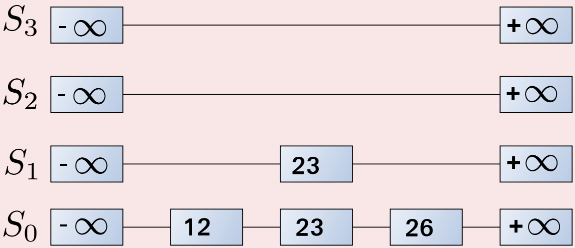
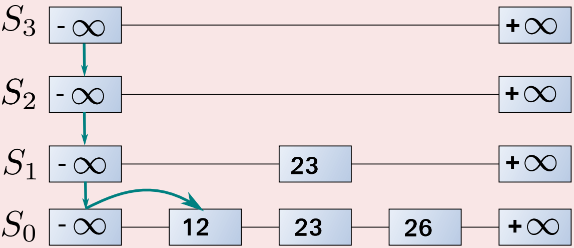
Example: Insertion of 15 in the skip list

First, we use $i = 2$ to insert S_3 into the skip list



Example: Insertion of 15 in the skip list

Clearly, you first search for the predecessor key!!!



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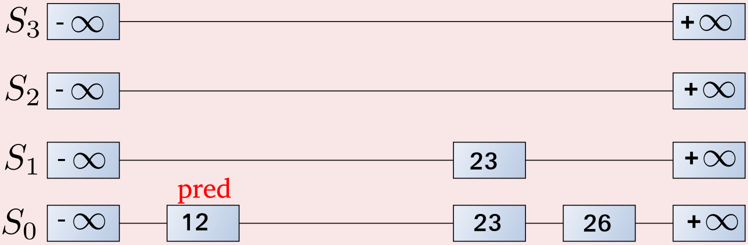
Insert the necessary Quad-Nodes and necessary information



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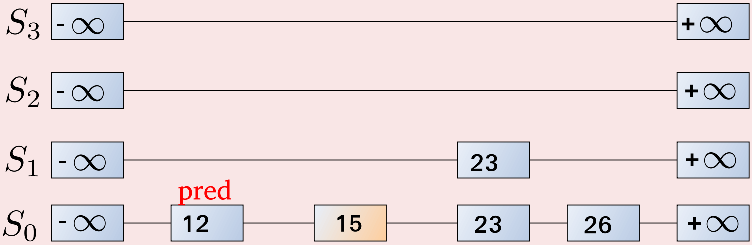

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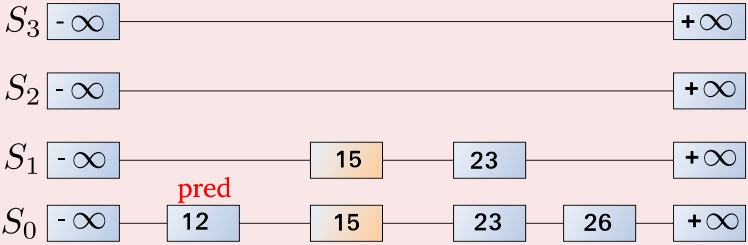

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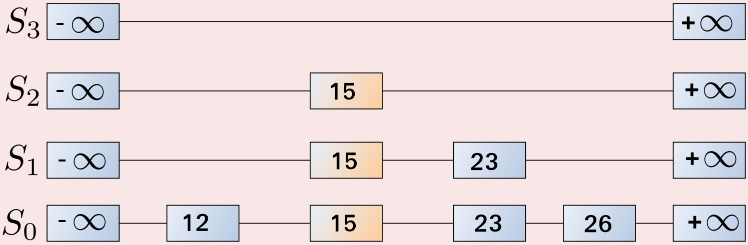
Finally!!!



Outline

Deletion

To remove an entry with key x from a skip list, we proceed as follows

- We search for x in the skip list and find the positions $p_0, p_1, ..., p_i$ of the items with key x, where position p_j is in list S_j.
- We remove positions $p_0, p_1, ..., p_i$ from the lists $S_0, S_1, ..., S_i$.
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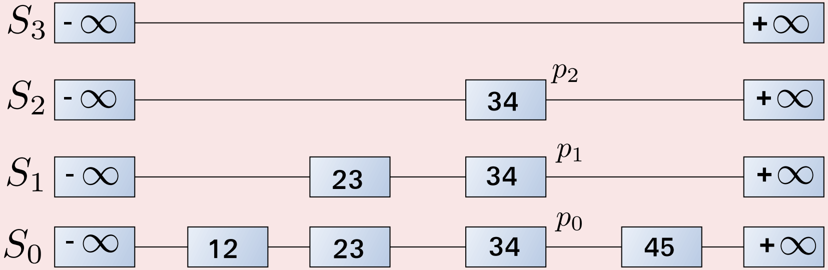
Example: Delete of 34 in the skip list

We search for 34 in the skip list and find the positions $p_0, p_1, ..., p_2$ of the items with key 34



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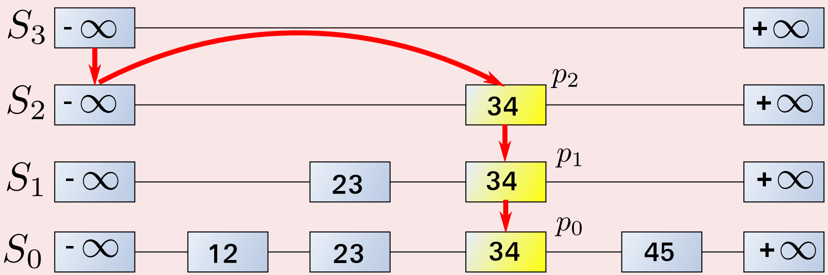
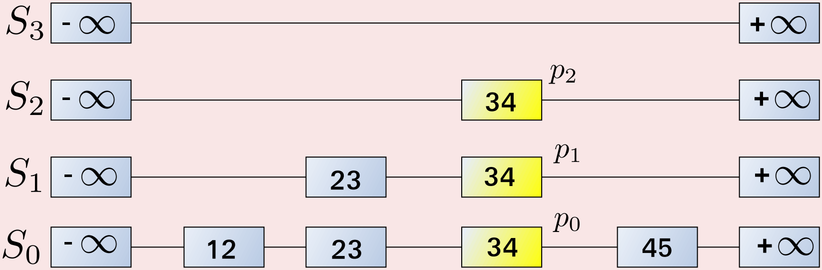


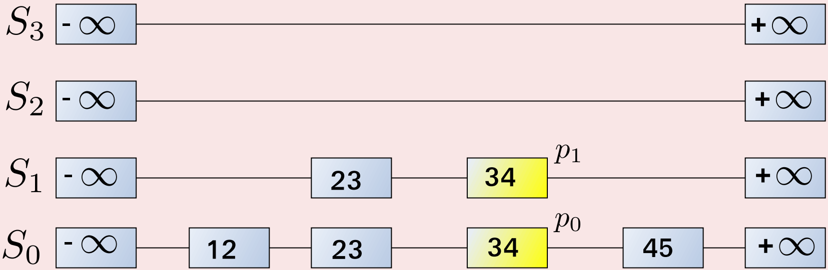
Example: Delete of 34 in the skip list

We start doing the deletion!!!



Example: Delete of 34 in the skip list

One Quad-Node after another



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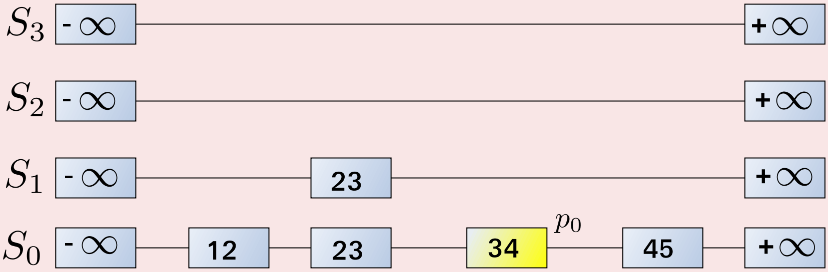

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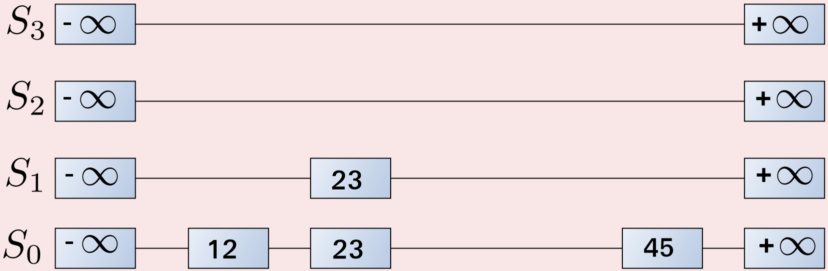

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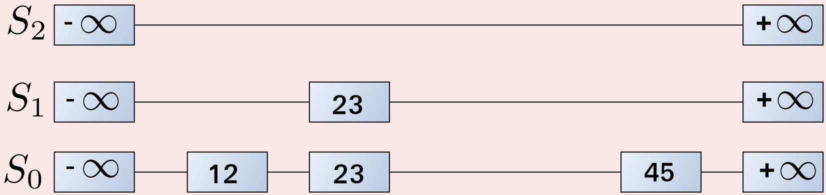
Remove One Level



Outline

Space usage

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The space used by a skip list depends on the random bits used by each invocation of the insertion algorithm.

Space : $O(n)$

Theorem

The expected space usage of a skip list with n items is $O(n)$.

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We use the following two basic probabilistic facts:

1. *Fact 1:* The probability of getting i consecutive heads when flipping a coin is $\frac{1}{2^i}$.
2. *Fact 2:* If each of n entries is present in a set with probability p, the expected size of the set is np.
   1. How? Remember $X = X_1 + X_2 + ... + X_n$ where X_i is an indicator function for event A_i = the i element is present in the set. Thus:

$$E[X] = \underbrace{\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} Pr\{A_i\}}_{\text{Equivalence } E[X_A] \text{ and } Pr\{A\}} = \sum_{i=1}^{n} p = np$$

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   1. How? Remember $X = X_1 + X_2 + ... + X_n$ where X_i is an indicator function for event A_i = the i element is present in the set. Thus:

$$E[X] = \underbrace{\sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} Pr\{A_i\}}_{\text{Equivalence } E[X_A] \text{ and } Pr\{A\}} = \sum_{i=1}^{n} p = np$$

Space : $O(n)$

Theorem

The expected space usage of a skip list with n items is $O(n)$.

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We use the following two basic probabilistic facts:

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Proof

Now consider a skip list with n entries

Using Fact 1, an element is inserted in list S_i with a probability of

$$P\left[x \in S_i\right] = \frac{1}{2^i}$$

Now by Fact 2

The expected size of list S_i is

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Proof

The expected number of nodes used by the skip list with height h

$$E\left[\text{Size Skip List}\right] = \sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i}$$

Here, we have a problem!!! What is the value of h?

Height h

First

The running time of the search and insertion algorithms is affected by the height h of the skip list.

Second

We show that with high probability, a skip list with n items has height $O(\log n)$.

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We use the following Fact 3

We can view the level $l(x_i) = \max\{j | \text{where } x_i \in S_j\}$ of the elements in the skip list as the following random variable

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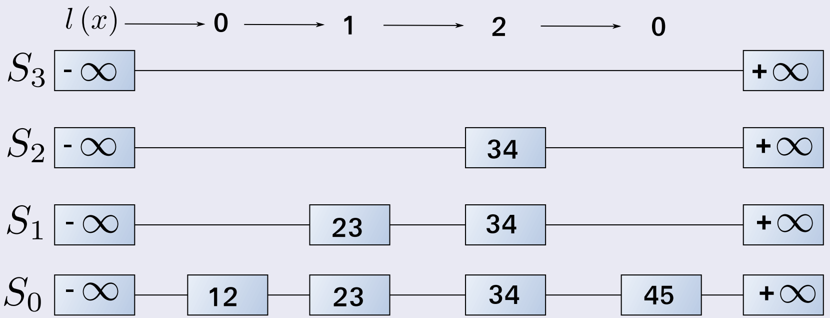
Example for $l\left(x_i\right)$

We have



BTW What is the geometric distribution?

k failures where

$$k = \{1, 2, 3, ...\}$$

Probability mass function

$$Pr(X = k) = (1 - p)^{k-1} p$$

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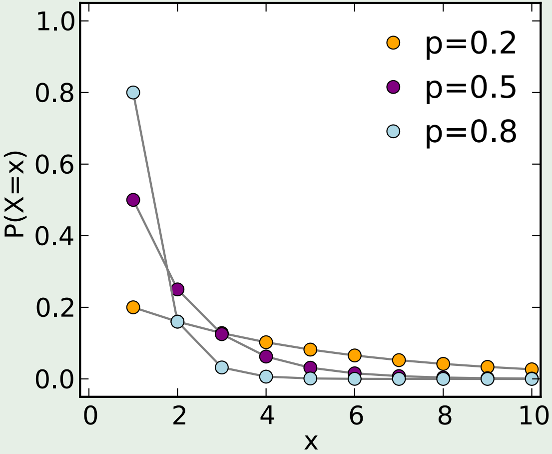
Probability Mass Function

For Different Probabilities



Then

We have the following inequality for the geometric variables

$$Pr\left[X_i > t\right] \leq (1-p)^t \;\; \forall i = 1, 2, ..., n$$

- If we assume we have a fair coin $p = \frac{1}{2}$

This is because

$$F(t) = P\left[X_i \leq t\right] = \sum_{i=1}^{t} (1-p)^{i-1} p$$

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\sum_{i=t+1}^{\infty} (1-p)^{i-1} p &= p \sum_{i=t+1}^{\infty} (1-p)^{i-1} \\
&= p \sum_{k=1,k=i-t}^{\infty} (1-p)^{k+t-1} \\
&= p\left(1-p\right)^{t} \sum_{k=1,k=i-t}^{\infty} (1-p)^{k-1} \\
&= (1-p)^{t} \frac{p}{1-p} \leq (1-p)^{t} \text{ Given the fair coin}
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Then, we have

Using our original formula

$$Pr\left[X_i > t\right] \leq (1-p)^t$$

In this way, we have

Then, we have

$$Pr\left\{\max_i X_i > t\right\} \leq n(1-p)^t$$

How?

We have that

$$
\begin{aligned}
Pr\left\{\max_i X_i > t\right\} &= Pr\left\{\max\left\{X_1, X_2, ..., X_n\right\} > t\right\} \\
&= \sum_{i=1}^{n} Pr\left\{X_i > t \text{ and } X_i = \max\left\{X_1, X_2, ..., X_n\right\}\right\}
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How?

- That one of the elements becomes the maximum in height and a height greater than t

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How?

- That one of the elements becomes the maximum in height and a height greater than t

Why?

Because the height of an element depends on independent event

- Each toss coin until tails is independent of the others!!!

Example

When having two lists

$$\{\max(X_1, X_2) > t\} = \{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t \text{ and } X_2 > X_1\}$$

- Yes, you need to remember that the max is a single element not both...

Therefore

Then

$$Pr\left\{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t\right\} = Pr\left\{X_1 > t \text{ and } X_1 > X_2\right\} + \dots$$
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Assuming exclusivity between phenomena $X_i > X_j$ and $X_i > t$

$$\begin{aligned} Pr\left\{X_1 > t \text{ and } X_1 > X_2 \text{ or exclusive } X_2 > t\right\} &= P\left(X_1 > t\right) P\left(X_1 > X_2\right) + \dots \\ & P\left(X_2 > t\right) P\left(X_2 > X_1\right) \\ &\leq P\left(X_1 > t\right) + P\left(X_2 > t\right) \end{aligned}$$

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Finally, using this fact

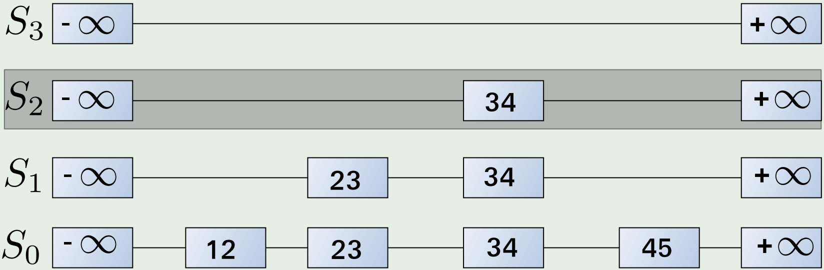
We have when summing over all events X_i

$$\sum_{i=1}^{n} Pr\left\{X_i > t\right\} \leq \sum_{i=1}^{n} \left(1-p\right)^t = n\left(1-p\right)^t$$

An Observation

The $\max_i X_i$

It represents the list with the one entry apart from the special keys.



Another One

Also REMEMBER!!!

We are talking about a fair coin, thus $p = \frac{1}{2}$.

Outline

Height: $3 \log_2 n$ with probability at least $1 - \frac{1}{n^2}$

Theorem

A skip list with n entries has height at most $3 \log_2 n$ with probability at least $1 - \frac{1}{n^2}$

Proof

Consider a skip list with n entires

By Fact 3, the probability that list S_t has at least one item (The $\max_i X_i > t$) is at most $\frac{n}{2^t}$.

$$P\left(|S_t| \geq 1\right) = P\left(\max_i X_i > t\right) \leq \frac{n}{2^t}.$$

By picking $t = 3\log n$

We have that the probability that $S_{3\log_2 n}$ has at least one entry is at most:

$$\frac{n}{2^{3\log_2 n}} = \frac{n}{n^3} = \frac{1}{n^2}.$$

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Finally

The expected number of nodes used by the skip list with height h

- Given that $h = 3\log_2 n$

$$\sum_{i=0}^{3\log_2 n} \frac{n}{2^i} = n \sum_{i=0}^{3\log_2 n} \frac{1}{2^i}$$

Given the geometric sum

$$S_m = \sum_{k=0}^{m} r^k = \frac{1 - r^{m+1}}{1 - r}$$

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We have finally

The Upper Bound on the number of nodes

$$n\sum_{i=0}^{3\log_2 n}\frac{1}{2^i} = n\left(\frac{1-\left(\frac{1}{2}\right)^{3\log_2 n+1}}{1-1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{2^{3\log_2 n+1}}}{1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{\left(2^{\log_2 n}\right)^3 2}}{1/2}\right)$$

$$= n\left(\frac{1-\frac{1}{2n^3}}{1/2}\right) = n\left(\frac{2\left[2n^3-1\right]}{2n^3}\right)$$

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Finally

We have

$$\left(\frac{2n^3 - 1}{n^2}\right) = 2n - \frac{1}{n^2} \leq 2n$$

The Upper Bound with probability $1 - \frac{1}{n^2}$

$$2n - \frac{1}{n^2} \leq 2n = O(n)$$

Finally

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$$\left(\frac{2n^3-1}{n^2}\right) = 2n - \frac{1}{n^2} \leq 2n$$

The Upper Bound with probability $1-\frac{1}{n^2}$

$$2n - \frac{1}{n^2} \leq 2n = O(n)$$

Outline

Search and Insertion Times

Fact 4

The expected number of coin tosses required in order to get tails is 2:

$$\text{Given that } x \sim G\left(\frac{1}{2}\right) \Longrightarrow E\left[x\right] = \frac{1}{p} = 2 \text{ (Fair Coin Assumption)}$$

We use this

To prove that a search in a skip list takes $O(\log n)$ expected time.

- After all insertions require searches!!!

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Search time

The search time in skip list is proportional to

the number of drop-down steps + the number of scan-forward steps

Drop-down steps

The drop-down steps are bounded by the height of the skip list and thus are $O(\log_2 n)$ with high probability.

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A search in a skip list takes $O(\log_2 n)$ expected time.

Proof

First

- When we scan forward in a list, the destination key does not belong to a higher list.

A scan-forward step is associated with a former coin toss that gave tails

- By Fact 4, in each list the expected number of scan-forward steps is 2.

Proof

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- When we scan forward in a list, the destination key does not belong to a higher list.

A scan-forward step is associated with a former coin toss that gave tails

- By Fact 4, in each list the expected number of scan-forward steps is 2.

Why?

Given the list S_i

- Then, the scan-forward intervals (Jumps between x_i and x_{i+1}) to the right of S_i are

$$I_1 = [x_1, x_2], I_2 = [x_2, x_3] \ldots I_k = [x_k, +\infty]$$

Then

These interval exist at level i if and only if all $x_1, x_2, \ldots, x_k$ belong to S_i.

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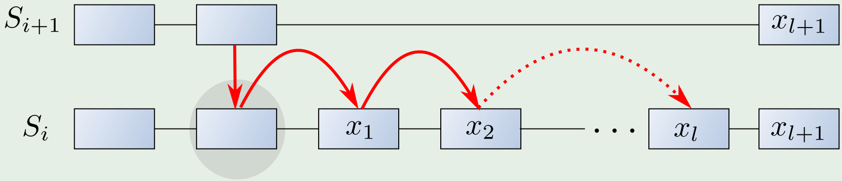
These interval exist at level i if and only if all $x_1, x_2, ..., x_k$ belong to S_i.

We introduce the following concept based on these intervals

Scan-forward siblings

These are element that you find in the search path before finding an element in the upper list.



Now

Given that a search is being done, S_i contains l forward siblings

It must be the case that given $x_1, ..., x_l$ scan-forward siblings, we have that

$$x_1, ..., x_l \notin S_{i+1}$$

and $x_{l+1} \in S_{i+1}$

Thus

We have

Since each element of S_i is independently chosen to be in S_{i+1} with probability $p = \frac{1}{2}$.

We have

The number of scan-forward siblings is bounded by a geometric random variable X_i with parameter $p = \frac{1}{2}$.

- Imagine the fact that you have multiple fails... then $x_1, ..., x_l \notin S_{i+1}$ is modeled by X_i

Thus, we have that

The expected number of scan-forward siblings is bounded by 2!!!

$$\text{Expected \# Scan-Forward Siblings at } i \leq \underbrace{E[X_i] = \frac{1}{1/2}}_{\text{Mean}} = 2$$

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In the worst case scenario

A search is bounded by $O(\log_2 n) + 2\log_2 n = O(\log_2 n)$

An given that a insertion is a (search) + (deletion bounded by the height)

Thus, an insertion is bounded by $2\log_2 n + 3\log_2 n = O(\log_2 n)$

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Applications

We have

- Cyrus IMAP servers offer a "skiplist" backend Data Base implementation.
- Lucene uses skip lists to search delta-encoded posting lists in logarithmic time.
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Thanks

Questions?