# Analysis of Algorithms

## Greedy Methods

Andres Mendez-Vazquez

February 19, 2018

# Outline

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# Steps of the greedy method

## Proceed as follows

- Determine the optimal substructure.
- Develop a recursive solution.
- Prove that at any stage of recursion, one of the optimal choices is the greedy choice.
- Show that all but one of the sub-problems resulting from the greedy choice are empty.
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# You are a thief

**You get into a store**

With a knapsack/bag

**The bag has capacity $W$**

You want to select items to fill the bag...

**Question**

How do you do it?

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# Greedy Process for 0-1 Knapsack

**First, You choose an ordering**

What about per price of each item?

If we have the following situation

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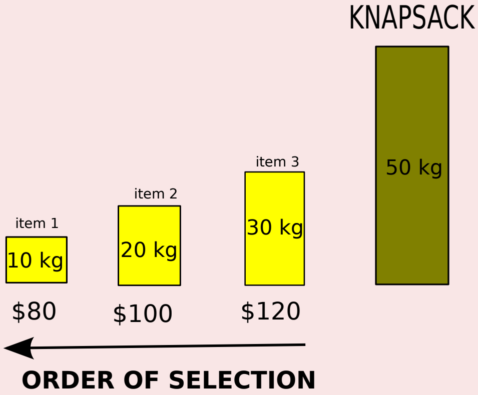

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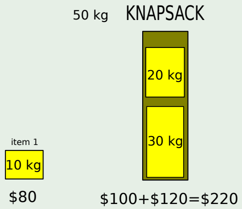


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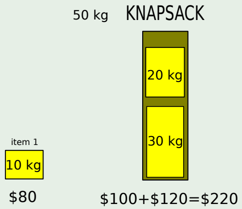


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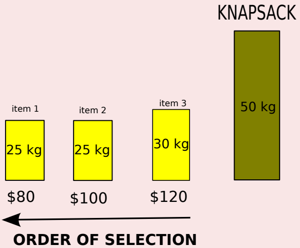

# Thus, we need a better way to select elements!!!

**Actually**

Why not to use the price of kg?

Thus, we have this!!!

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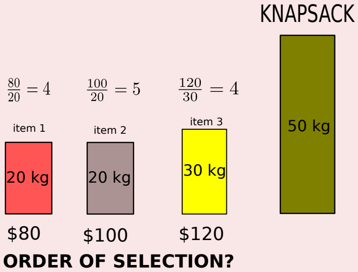

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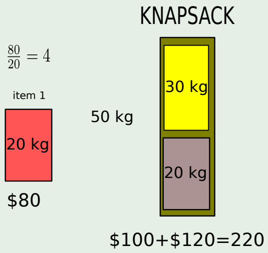

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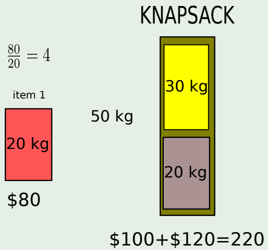


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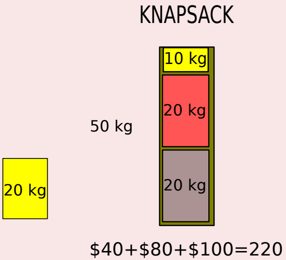
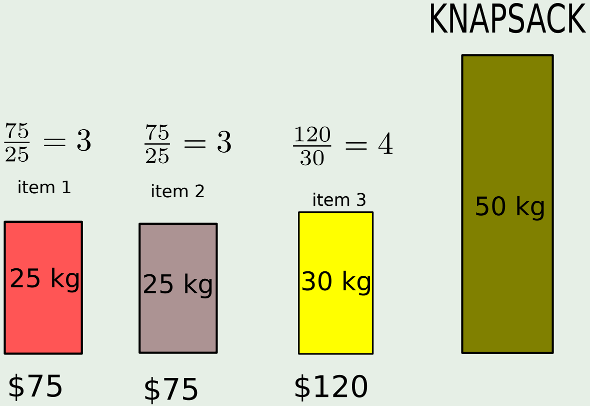


# However!!!

## Even with an order based in $\frac{v}{w}$ 0-1 Knapsack fails!!!




**ORDER OF SELECTION?**

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# Theorem about Greedy Choice

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The greedy choice, which always selects the object with better ratio value/weight, always finds an optimal solution to the Fractional Knapsack problem.

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Constraints:

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**FRACTIONAL-KNAPSACK$(W, w, v)$**

1. for $i = 1$ to n do $x[i] = 0$
2. $weight = 0$
3. // Use a Max-Heap
4. T = Build-Max-Heap$(v/w)$
5. while $weight < W$ do
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7. if $(weight + w[i] \leq W)$ do
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# Activity selection

## Problem

Set of activities $S = a_1, ..., a_n$. The $a_i$ activity needs a resource (Class Room, Machine, Assembly Line, etc) during period $[s_i, f_i)$, which is a half-open interval, where $s_i$ is the start time of activity $a_i$ and $f_i$ is the finish time of activity $a_i$.

## For example

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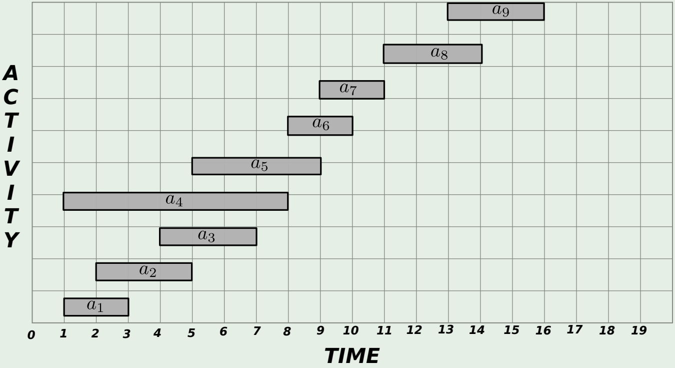


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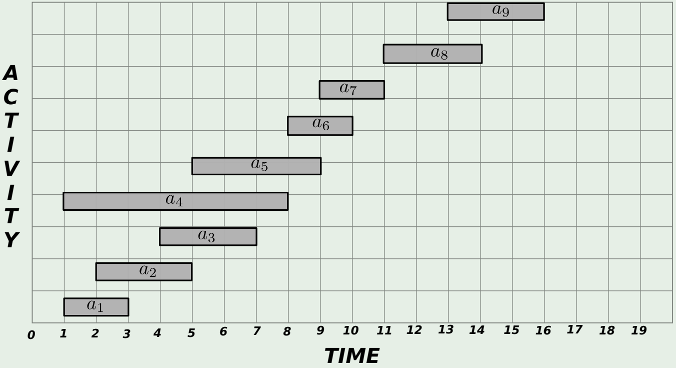


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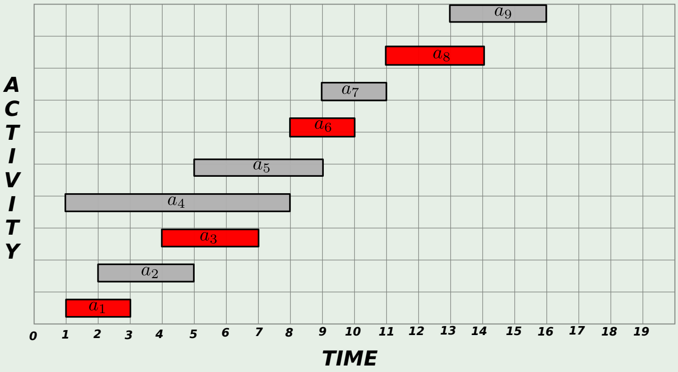

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We need the recursive solution!!!

**Third**

The Greedy Choice

**Fourth**

Prove it is the only one!!!

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**We can then refine the set of activities in the following way**

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- $S_{ij}$ = the set of activities that start after activity $a_i$ finishes and that finishes before activity $a_j$ start.

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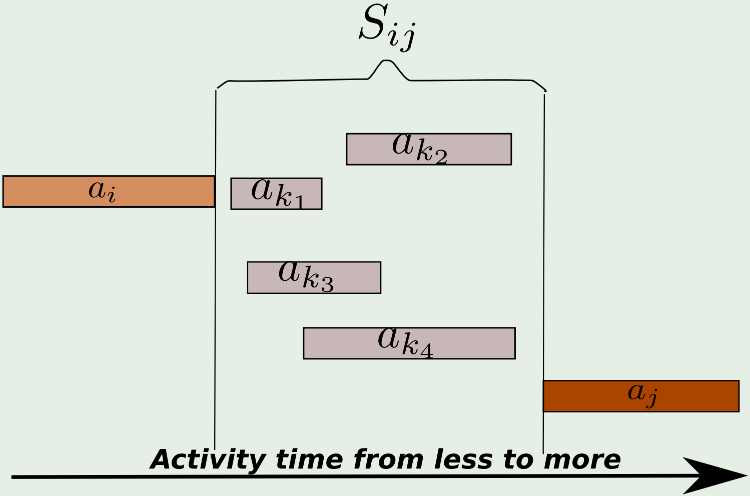


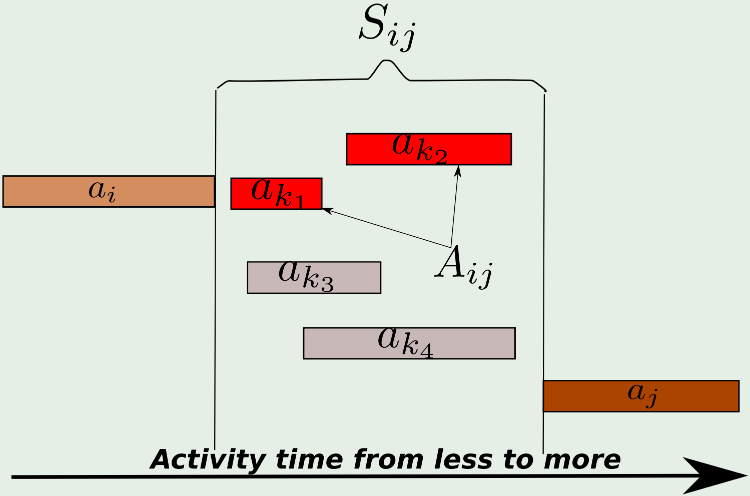
# The Optimal Substructure

## Second

- Suppose that the set $A_{ij}$ denotes the maximum set of compatible activities for $S_{ij}$

# For Example

## Thus, we have



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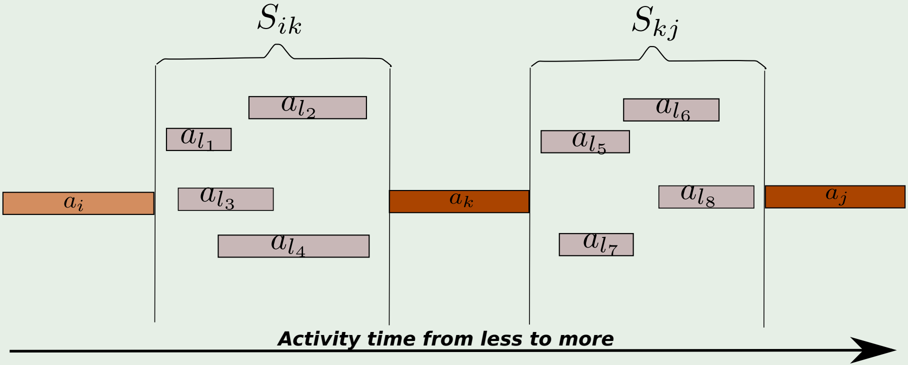

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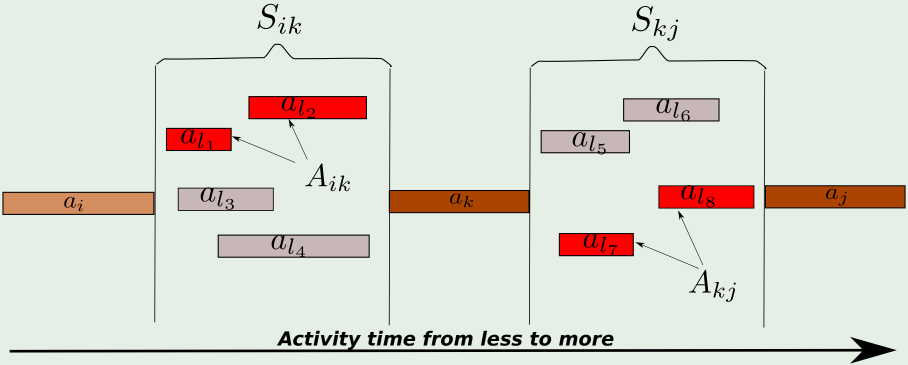

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## Use Cut-and-Paste Argument

How? Assume that exist $A_{ik}^{'}$ such that $\left|A_{ik}^{'}\right| > |A_{ik}|$.

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$\neg P =$exist $A^{'}_{ik}$ such that $\left|A^{'}_{ik}\right| > |A_{ik}|$.

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**This means that $A'_{ij}$ has more activities than $A_{ij}$**

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Can anybody give me the recursion?

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$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \varnothing \\ \max\limits_{\substack{i<k<j \\ a_k \in S_{ij}}} c[i,k] + c[k,j] + 1 & \text{if } S_{ij} \neq \varnothing \end{cases}$$

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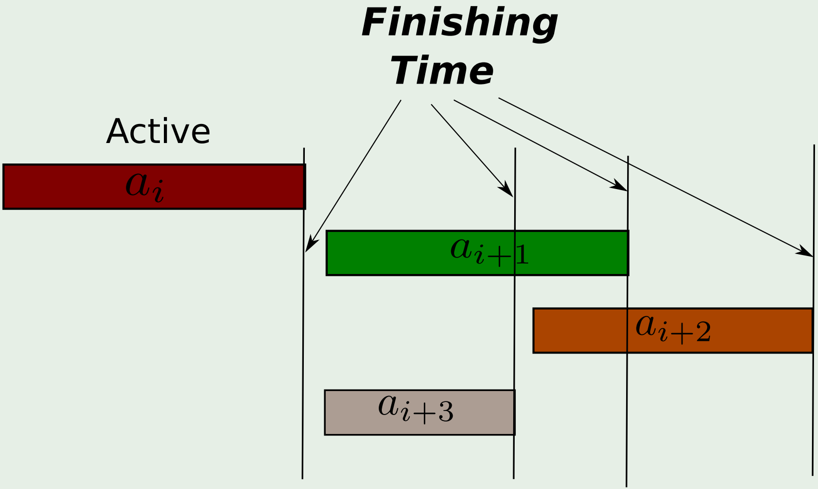

# The Early Finishing Time

**Yes!!!**

The Greedy Choice = **Select by earliest finishing time**.

# Greedy solution

**Theorem 16.1**

Consider any nonempty subproblem $S_k = \{a_i \in S | s_i > f_k\}$, and let $a_m$ be an activity in $S_k$ with the earliest finish time. Then $a_m$ is included in some maximum-size subset of mutually compatible activities of $S_k$ .

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An algorithm to solve the activity-selection problem does not need to work bottom-up!!!

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It can work top-down, choosing an activity to put into the optimal solution and then solving the subproblem of choosing activities from those that are compatible with those already chosen.

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The activities are already sorted by finishing time

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There are dummy activities:

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**REC-ACTIVITY-SELECTOR$(s, f, k, n)$ - Entry Point $(s, f, 0, n)$**

1. $m = k + 1$
2. // find the first activity in $S_k$ to finish
3. while $m \leq n$ and $s[m] < f[k]$
4. $\quad m = m + 1$
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# Do we need the recursion?

**Did you notice?**

We remove all the activities before $f_k$!!!

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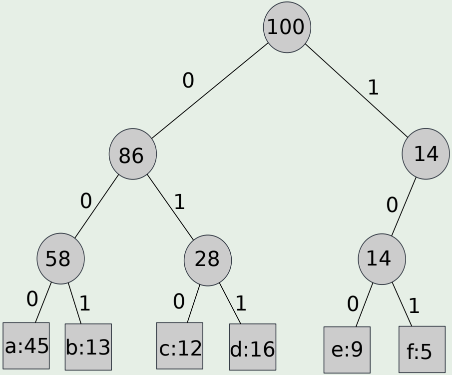

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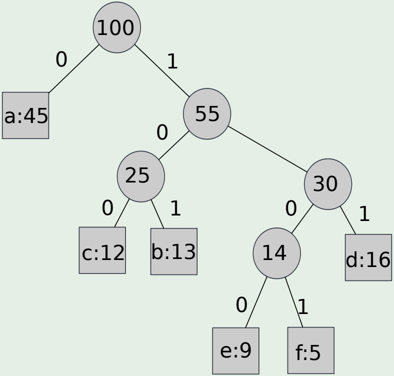

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# Cost of Trees for Coding

**Cost in the number of bits for a text**

$$B(T) = \sum_{c \in C} c.freq \times d_T(c).$$

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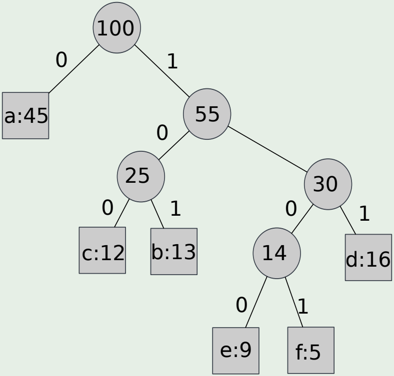

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## Process

- You start with an alphabet $C$ with an associated frequency for each element in it.
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- Build a three using as children the two nodes of the subtrees extracted from the min priority queue. The new root holds the sum of frequencies of the two subtrees.
- Put it back into the Priority Queue.

# Greedy Process for Huffman Codes

## Process

- You start with an alphabet $C$ with an associated frequency for each element in it.
- Use the frequencies to build a min priority queue.
- Subtract the two least frequent elements (Greedy Choice).
- Build a three using as children the two nodes of the subtrees extracted from the min priority queue. The new root holds the sum of frequencies of the two subtrees.
- Put it back into the Priority Queue.

# Algorithm

**HUFFMAN($C$)**

$n = |C|$
Q = C
for $i$ = 1 to $n$-1
    allocate new node $z$
    z.left = x = Extract-Min(Q)
    z.right = y = Extract-Min(Q)
    z.freq = x.freq+y.freq
    Insert(Q,z)
return Extract-Min(Q) // return root of the Huffman Tree

**Complexity**

$$\Theta(n \log n)$$

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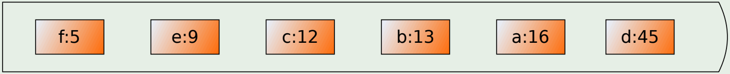
# Example

## The Process!!!



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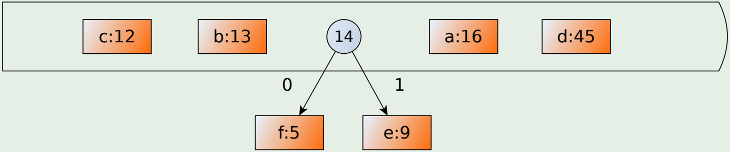

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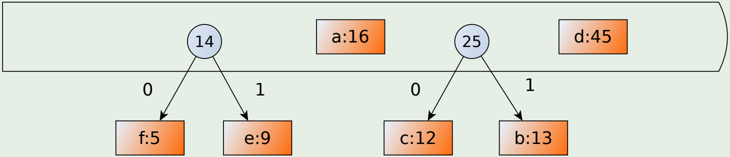

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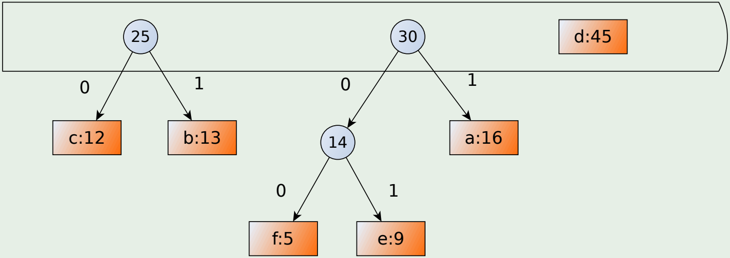

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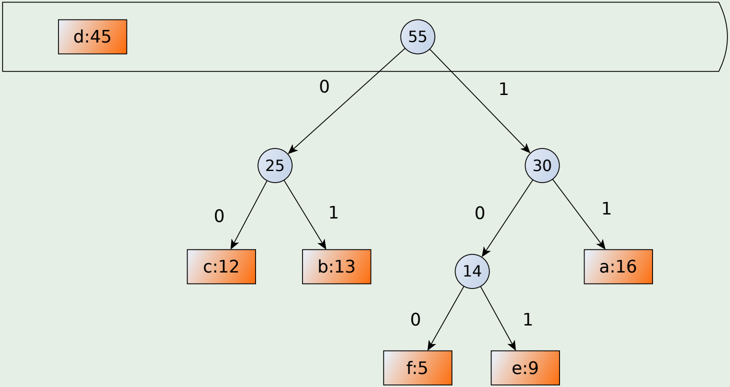

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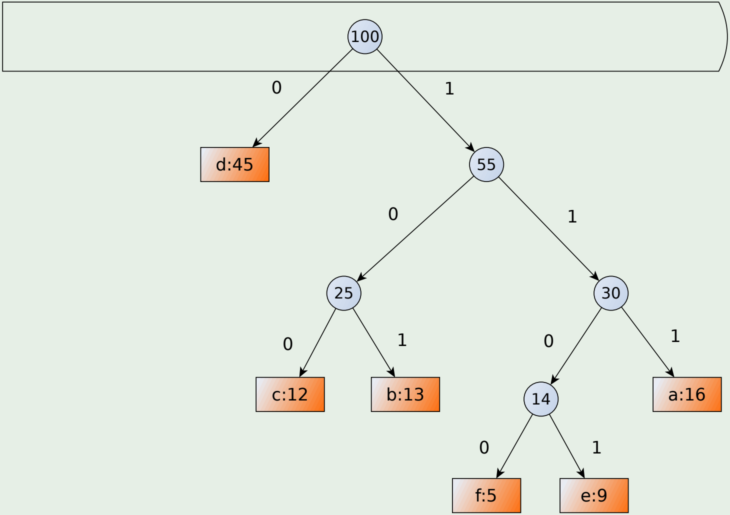

# Lemmas to sustain the claims

**Lemma 16.2**

Let $C$ be an alphabet in which each character $c \in C$ has frequency $c.freq$. Let $x$ and $y$ be two characters in $C$ having the lowest frequencies. Then there exists an optimal prefix code for $C$ in which the codewords for $x$ and $y$ have the same length and differ only in the last bit.

**Lemma 16.3**

Let $C$ be a given alphabet with frequency c:freq defined for each character $c \in C$ . Let $x$ and $y$ be two characters in $C$ with minimum frequency. Let $C'$ be the alphabet $C$ with the characters $x$ and $y$ removed and a new character $z$ added, so that $C' = C - \{x, y\} \cup \{z\}$. Define $f$ for $C'$ as for $C$, except that $z.freq = x.freq + y.freq$. Let $T'$ be any tree representing an optimal prefix code for the alphabet $C'$. Then the tree $T$, obtained from $T'$ by replacing the leaf node for $z$ with an internal node having $x$ and $y$ as children, represents an optimal prefix code for the alphabet $C$.

# Lemmas to sustain the claims

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# Exercises

## From Cormen's book solve the following

- 16.2-1
- 16.2-4
- 16.2-5
- 16.2-7
- 16.1-3
- 16.3-3
- 16.3-5
- 16.3-7