Analysis of Algorithms

Dynamic Programming

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Outline

1. Dynamic Programming
   - Bellman Equation
   - Elements of Dynamic Programming
   - Rod Cutting
2. Elements of Dynamic Programming
   - Optimal Substructure
   - Overlapping Subproblems
   - Reconstruction of Subproblems
   - Common Subproblems
3. Examples
   - Longest Increasing Subsequence
   - Matrix Multiplication
   - Longest Common Subsequence
4. Exercises

History

Dynamic Programming

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Bellman Equation

Definition

$$V(x_0) = \max_{a_0}[F(x_0) + \beta V(x_1)]$$

$$\text{s.t. } a_0 \in \Gamma(x_0), x_1 = T(x_0, a_0)$$

- Where $\Gamma(x_0)$ is a set of actions depend on the current state.
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Looks Terrifying!!!

However

It is quite simple!!!

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Elements of Dynamic Programming

Define the Optimal Structure

Characterize the structure of an optimal solution.

Define the Recursion

Recursively define the value of an optimal solution.

Compute the Solution

Compute the value of an optimal solution, typically bottom-up.

IMPORTANT!!!

We use an extra memory to stop the recursion!!!

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Elements of Dynamic Programming

Finally Rebuild the Optimal Solution

Construct an optimal solution from computed information.

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Rod cutting

Problem

Given a rod of length n inches and a table of prices p_i for $i = 1, 2, ..., n$, determine the maximum revenue r_n obtainable by cutting up the rod and selling the pieces.

Rod Cutting table

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

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Characterize the structure of an optimal solution

Example

For example for a rod of size 10, we could cut the rod in 3 parts, $10=4+3+3$.

Thus

Then, we can assume that an optimal solution cuts the rod in k pieces, $1 \leq k \leq n$ i.e. $k-1$ cuts.

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The length of each piece can be numbered as

i_j with $1 \leq j \leq k$

The total size of the rod is then

Thus, the max revenue

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$$n = i_1 + i_2 + ... + i_k$$

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$$r_n = p_{i_1} + p_{i_2} + ... + p_{i_k}$$

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For length $n = 4$ by brute force approach

1. price equal to 9
2. price equal to 1+8
3. price equal to 8+1
4. price equal to 1+1+5
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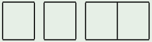
4.  price equal to 1+1+5
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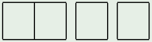
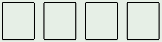
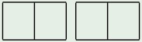
6.  price equal to 5+1+1
7.  price equal to 1+1+1+1
8.  **price equal to 5+5 Optimal!!!**

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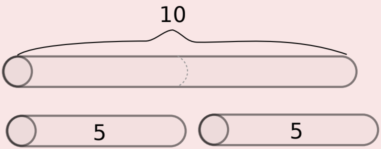

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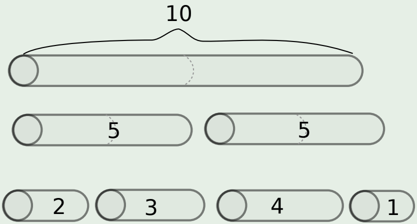

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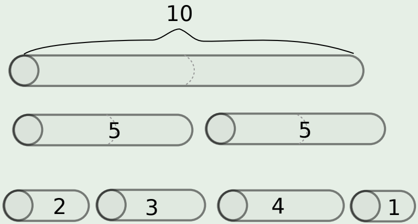

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Thus, we take a final decision!!!

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Which One?

The Largest One

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$$r_n = \max\left\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, ..., r_{n-1} + r_1\right\}$$

Some stuff about the optimal solution

Did you notice the following?

Once you get an optimal solution!!! The Most Revenue!!!

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Formally: Cut and Paste

Given

$n = i_1 + i_2 + ... + i_k$

Imagine, we split the problem in two parts

$A_1 = \{i_1, i_2, ..., i_l\}$ and $A_2 = \{i_{l+1}, i_2, ..., i_k\}$

Properties

Now imagine that exist a $A_1' = \left\{i_1', i_2', ..., i_l'\right\}$ such that:

$$r_n' = p_{i_1'} + p_{i_2'} + ... + p_{i_l'} > r_n = p_{i_1} + p_{i_2} + ... + p_{i_l}$$

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Then, we have a set of cuts

$A_1' \cup A_2$ with better revenue than the original cut-set!!!

Clearly

Contradiction!!!

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So we need to convert this into something more programmable

You can define Cut-Rod$(p, n - i)$ where

- p is an array with the table values.
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Finally

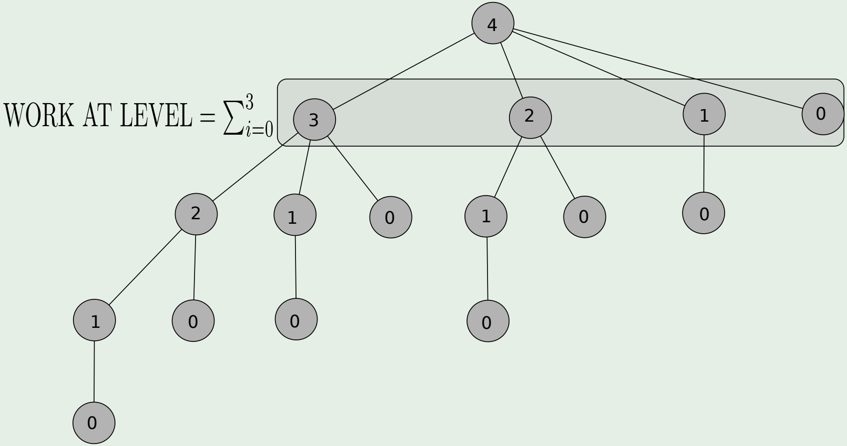
Code

Cut-Rod(p, n)

1. if $n == 0$
2. return 0
3. $q = -\infty$
4. for $i = 1$ to n
5. $q = \max\{q, p[i] + \text{Cut-Rod}(p, n - i)\}$
6. return q

How the recursion tree for this code looks like?

First, Did you notice this?



Recursion

We have finally

$$T(n)=\begin{cases}1 & \text{if } n=0\\ 1+\sum_{j=0}^{n-1}T(j) & \text{if } n>0\end{cases} \tag{1}$$

- 1 for calling into the root of the tree.
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How many possible decisions are being considered when cutting?

Decision	cut at 1	cut at 2	$\cdots$	cut at n-1
Which One?	0 or 1	0 or 1	$\cdots$	0 or 1

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What the tree is telling us?

The number of possible paths is equal to the number of leaves

- We have 2^{n-1} paths, which is equal to the number of leaves

Then

- The recursion consider explicitly all possible decisions

It is possible to prove by induction that

$$T(n) = 2^n \tag{2}$$

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First Approach: Top-down with Memoization

Basics in this approach

1. We write the procedure recursively in a natural manner.
2. However, we save the result of each subproblem (Usually in an array or hash table)

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1. We write the procedure recursively in a natural manner.
2. However, we save the result of each subproblem (Usually in an array or hash table)

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We require an Auxiliary Function to Accomplish this

Code

Memoized-Cut-Rod(p, n)

1. Let $r[0..n]$ be a new array
2. for $i = 0$ to n
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3. if $n == 0$
4. $q = 0$
5. else $q = -\infty$
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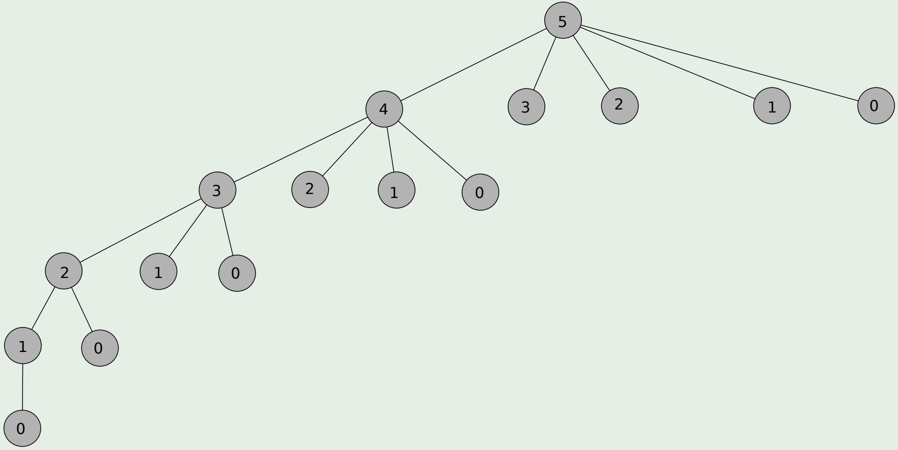
The Recursion Tree of Memoized-Cut-Rod

Tree for $n = 5$



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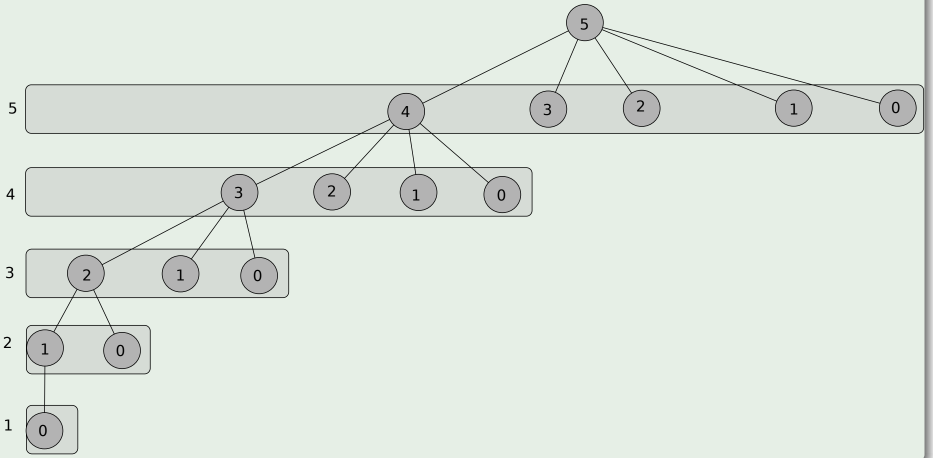
Then look at this..

Something Notable



Complexity

Add the works

We have then

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} \quad (3)$$

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What about the Bottom-Up approach?

Simpler Solution

How?

The natural order of solving

A problem of size i is smaller than a subproblem of size j, if $i < j$.

It is simpler to solve problems in this orden

$j = 0, 1, 2, ..., n$ in order of increasing size.

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How to See Everything: Subproblem Graphs (DAG)

In dynamic programing

It is necessary to understand how subproblems depend on each other.

This information can be found in the subproblem graph which is a DAG

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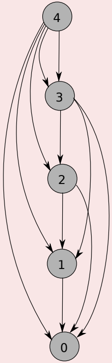

Reconstructing the Solution

How, we can do that?

Any Ideas?

We need to...

So...

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Store each **choice** of the solution some way

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We can reconstruct the solution path

Final Code

Code

Extended-Bottom-Up-Cut-Rod(p, n)

1. Let $r[0..n]$ and $s[0..n]$ be new arrays
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6. $\quad\quad$ if $q < p[i] + r[j-i]$
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Printing Code

Code

Print-Cut-Rod-Solution(p, n)

1. (r, s) =Extended-Bottom-Up-Cut-Rod(p, n)
2. while $n > 0$
3. print $s[n]$
4. $n = n - s[n]$

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Example

From the previous problem

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Thus

i	0	1	2	3	4	5	6	7	8	9	10
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Outline

Optimal Substructure

In dynamic programming

A first step toward the solution is characterizing the problem and finding the optimal substructure.

We have the following steps

First

The problem consists in making choices.

Second

Given each problem, you are given a choice that leads to a solution.

Third

Each solution allows us to determine which subproblems need to be solved, and how to best characterize the resulting space of subproblems.

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Fourth

Use cut-and-paste to prove by contradiction that the optimal subproblem structure exists.

Now using the following problems

Unweighted shortest path

Find a path from u to v consisting of the fewest edges.

Unweighted longest simple path

Find a simple path from u to v consisting of the most edges.

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Unweighted shortest path

It has an optimal substructure

Why?

First, given an optimal shortest path t between p and q.

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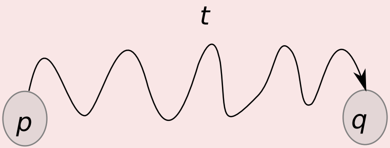
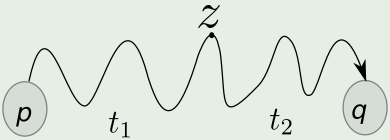


How do we prove this?

First

Assume an intermediate point z such that there are two paths t_1 and t_2, $t = t_1 \cup t_2$



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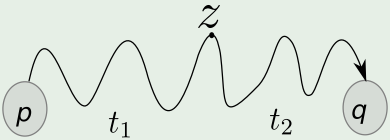

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Some problems do not have the optimal substructure

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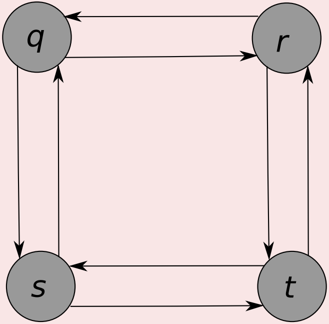

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First: Possible path between q and t

$$q \longrightarrow r \longrightarrow t$$

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$q \longrightarrow r$ is not the longest simple path from q and r nor the path $r \longrightarrow t$

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Then, How can we use the DAG?

Get the Space Problem

- Use the elements of the space.
- Build a Graph using all the decisions that can be made.
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What is the difference?

In the Unweighted Shortest Path the problems are independent

We mean that the solution to one sub-problem does not affect the solution of another subproblem.

In the Unweighted Longest Path

Remember vertices q and r in the second case!!!

Question

Then, Why the USP are independent?

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Overlapping Subproblems

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This happens because the recursive solution revisits the same subproblem multiple times.

This is the main advantage of dynamic programming

It takes advantage of this by solving and storing the solution.

Properties

A dynamic-programming solution runs in polynomial time when the number of distinct subproblems involved is polynomial in the input size and they can be solved in polynomial time.

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Overlapping Subproblems

We have two ways of solving the problem

- Top-down with Memoization.
- Bottom-up.

Outline

Reconstruction of Subproblems

To reconstruct

We use a table to store the choices such that we can reconstruct those of the sub-problem.

Outline

Common Subproblems

Something Notable

Finding the right subproblem takes creativity and experimentation.

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The input is $x_1, x_2, ..., x_n$.

Subproblems

$x_1, x_2, ..., x_i$

Example

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The number of subproblems is therefore linear.

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$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Therefore

The number of subproblems is therefore linear.

Number of Subproblems is Linear

We have the following input

The input is $x_1, x_2, ..., x_n$.

Subproblems

$x_1, x_2, ..., x_i$

Example

$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

Therefore

The number of subproblems is therefore linear.

Number of Subproblems is $O(nm)$

Input

The input is $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_m$.

Subproblems

$x_1, x_2, ..., x_i$ and $y_1, y_2, ..., y_j$.

Example

$$\boxed{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

$$\boxed{y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5} \quad y_6 \quad y_7 \quad y_8$$

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Number of Subproblems is $O(nm)$

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The number of subproblems is $O(mn)$.

Number of Subproblems is $O\left(n^2\right)$

Input

The input is $x_1, x_2, ..., x_n$.

Subproblems

$x_i, x_{i+1}, \ldots, x_j$

Example

$$x_1 \quad x_2 \quad \boxed{x_3 \quad x_4 \quad x_5 \quad x_6} \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

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Input is a rooted subtree

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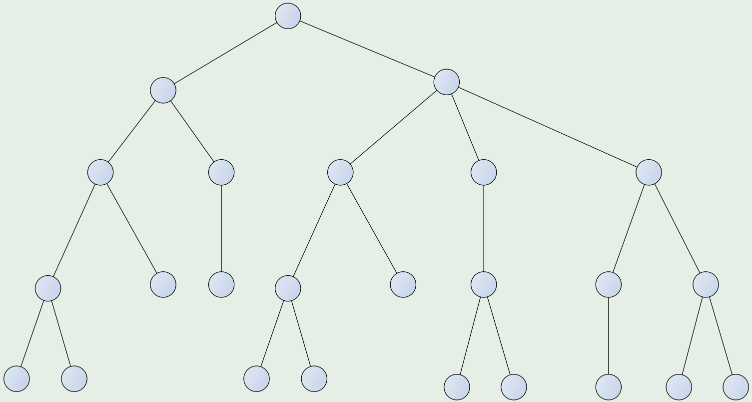
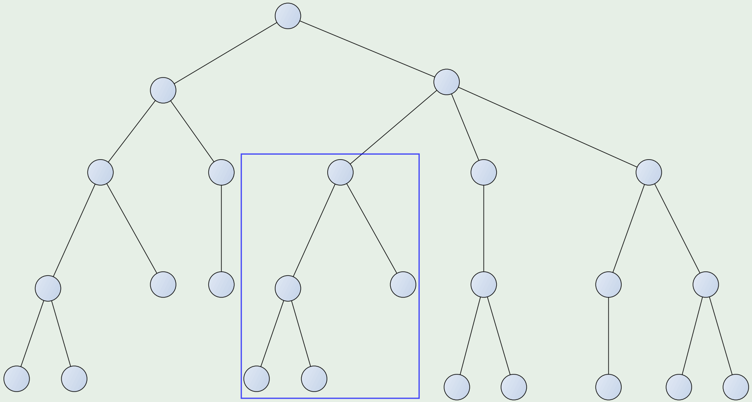


Input is a rooted subtree

Subproblem



Question

How Many Subproblems do you have?

Any Idea?

Outline

Definition

Input

A sequence $a_1, a_2, ..., a_n$

A subsequence

It is any subset of these numbers taken in order $a_{i_1}, a_{i_2}, ..., a_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Thus

An increasing subsequence is one in which the numbers are getting strictly larger.

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The task is to find the increasing subsequence of greatest length.

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Example

5 2 8 6 3 6 9 7

The Graph of increasing subsequences

To better understand the solution space, we can create the graph of all permissible transitions

- First, establish a node i for each element a_i, and add directed edges (i, j) whenever possible.
- i.e. Whenever $i < j$ and $a_i < a_j$.

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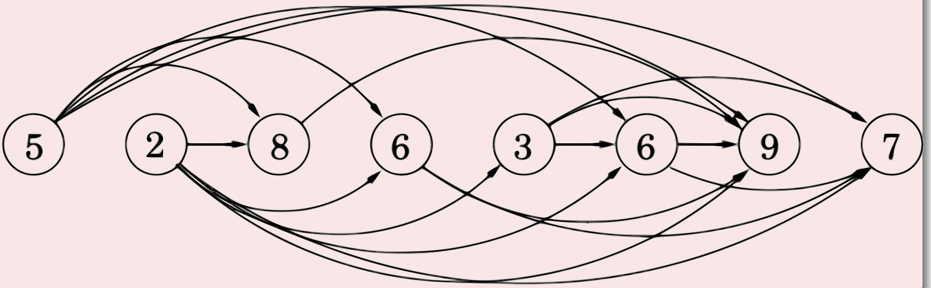

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- Thus, find the longest path in the DAG.

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Formulation

Something Notable

If we choose a number a_j to be in the longest increasing subsequence

We ask if the there is an edge to another

Is $(i, j) \in E$?

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Thus

We start at a certain j

Then, we look at the previous i with $1 \leq i \leq j-1$

Here is the recursion for $\forall A[i] < A[j]$

$$L[j] = \begin{cases} 1 & \text{if there is no edge } (i,j) \in E \\ 1 + \max\{L[i_1], L[i_2], ..., L[i_h]\} & \text{For } (i_k, j) \in E, 1 \leq k \leq h \end{cases} \tag{4}$$

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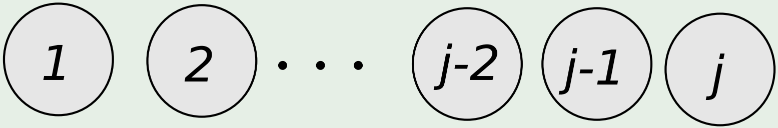
What is the meaning of this?

When is there an edge between i_k and j?



Clearly, this needs to be implemented in a machine

We have then that

A is an array that contains numbers indexed from 1 to n

Then, we have that

Instead of using $(i_k, j) \in E$ we use $A[i_k] < A[j]$

Instead of max

We use a loop and something like $q < temp$ for it

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Recursive Function

The final recursive code

Recursive-Longest-Subsequence(A, n)

1. $q = 1$

Recursive Function

The final recursive code

Recursive-Longest-Subsequence(A, n)

1. $q = 1$
2. // Assume n as part of your solution
3. // Thus $A[i] < A[n]$ here $j == n$
4. for $i = 1$ to $n - 1$
5. $\quad t =$Recursive-Longest-Subsequence(A, i)
6. $\quad$ if $A[i] < A[n]$ and $q < 1 + t$
7. $\quad\quad q = 1 + t$
8. return q

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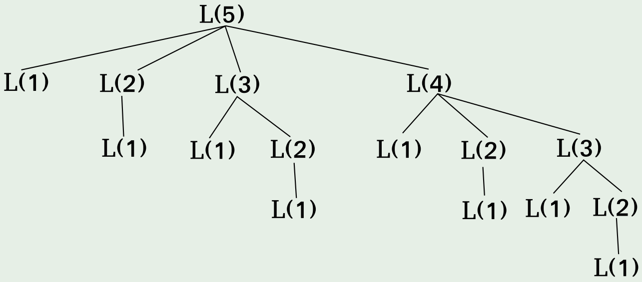
What about the Complexity?

Recursion Tree - Can somebody Guess the Complexity?



How we save in recursive calls

First

Let $L[1..n]$ an array to store the values the longest subsequence

Bottom-Up Solution

Code

Bottom-Up-Longest-Subsequence(A, n)

```
Let L[1..n]
max = 0
for i = 1 to n
        L[i] = 1
for j = 2 to n
        for i = 1 to j − 1
                if A[i] < A[j] and L[j] < L[i] + 1
                        L[j] = L[i] + 1
for i = 1 to n
        if max < L[i]
                max = L[i]
return max
```

Step 1

- An array to store the values the longest subsequence.

Bottom-Up Solution

Code

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Let L[1..n]
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for i = 1 to n
        if max < L[i]
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return max
```

Step 2

- A measure about the longest subsequence.

Bottom-Up Solution

Code

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Let $L[1..n]$
$max = 0$
for $i = 1$ to n
  $L[i] = 1$
for $j = 2$ to n
  for $i = 1$ to $j - 1$
    if $A[i] < A[j]$ and $L[j] < L[i] + 1$
      $L[j] = L[i] + 1$
for $i = 1$ to n
  if $max < L[i]$
    $max = L[i]$
return max

Step 3

- Initialize everything to 1 (Itself).

Bottom-Up Solution

Code

Bottom-Up-Longest-Subsequence(A, n)

Let $L\left[1..n\right]$
$max = 0$
for $i = 1$ to n
    $L\left[i\right] = 1$
for $j = 2$ to n
    for $i = 1$ to $j - 1$
        if $A[i] < A[j]$ and $L\left[j\right] < L\left[i\right] + 1$
            $L\left[j\right] = L\left[i\right] + 1$
for $i = 1$ to n
    if $max < L\left[i\right]$
        $max = L\left[i\right]$
return max

Step 4

- We know that the subproblem with size 1 has a solution, thus you need to start at 2.

Bottom-Up Solution

Code

Bottom-Up-Longest-Subsequence(A, n)

1. Let $L\left[1..n\right]$
2. $max = 0$
3. for $i = 1$ to n
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6. $\quad$ for $i = 1$ to $j - 1$
7. $\quad\quad$ if $A[i] < A[j]$ and $L\left[j\right] < L\left[i\right] + 1$
8. $\quad\quad\quad L\left[j\right] = L\left[i\right] + 1$
9. for $i = 1$ to n
10. $\quad$ if $max < L\left[i\right]$
11. $\quad\quad max = L\left[i\right]$
12. return max

Step 5

- Get solutions to the subproblems less or equal than $j - 1$

Bottom-Up Solution

Code

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Step 6

- Take a decision

Bottom-Up Solution

Code

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                        L[j] = L[i] + 1
for i = 1 to n
        if max < L[i]
                max = L[i]
return max
```

Step 7

- If the decision is true then increase the counter for solution starting at j

Bottom-Up Solution

Code

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for i = 1 to n
    if max < L[i]
        max = L[i]
return max
```

Step 8

- Find the Maximum Value

What about backtracking the Solution

We can do the following

You can have an array $S[1..n]$ initialized to the sequence $1, 2, ..., n$

Thus, each time

$A[i] < A[j]$ and $L[j] < L[i] + 1$ is true, we set $S[j] = i$.

Then

After returning the L and S we can get the index of the max to backtrack the answer.

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Outline

Definition of The Problem

Input

A sequence of Matrices $\langle A_1, A_2, ..., A_n \rangle$

Output

We want a fully parenthesized product, where the final result is a single matrix or the product of two fully parenthesized matrix products.

Why

Take in consideration the following algorithm

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Why? Look at this pseudocode

MATRIX-MULTIPLY(A,B)

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if A.columns ≠ B.rows
        error "incompatible dimensions"
else let C be a new A.rows × B.columns matrix
            for i = 1 to A.rows
                for for j = 1 to B.columns
                    c_ij = 0
                    for k = 1 to A.columns
                        c_ij = c_ij + a_ik · b_kj
return C
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MATRIX-MULTIPLY(A,B)

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Then

If A is $N \times M$ and B is a $M \times P$ then the cost is $N \cdot M \cdot P$.

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Example of Matrix Multiplications

Given the following matrices

- A, B, C with 10×100, 100×5 and 5×50
- Cost in scalar operations of (AB) is $10 \cdot 100 \cdot 5 = 5000$
- Cost in scalar operations of (BC) is $100 \cdot 5 \cdot 50 = 25000$

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Cost in scalar operations of $(AB)C$ is $5000 + 10 \cdot 5 \cdot 50 = 7500$

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Cost in scalar operations of $(AB)C$ is $5000 + 10 \cdot 5 \cdot 50 = 7500$
Cost in scalar operations of A(BC) is $25000 + 10 \cdot 100 \cdot 50 = 75000$

Matrix-Chain Multiplication

Problem

Given a chain $\langle A_1, A_2, ..., A_n \rangle$ of n matrices, where A_i has dimension $p_{i-1} \times p_i$. We want to fully parenthesize the product $A_1 A_2 ... A_n$ to minimize the number of scalar multiplications

Solving by brute force

Count all the possible parenthesizations

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \end{cases}$$

Which is the sequence of Catalan Numbers which grows

$$\Omega\left(\frac{4^n}{n^{\frac{3}{2}}}\right)$$

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In addition

Look at the following multiplication

$$(A_i \cdots A_k)\,(A_{k+1} \cdots A_j)$$

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The total cost of this multiplication is

$m[i,k] + m[k+1,j] + p_{i-1}p_kp_j$ **(In addition, you want to minimize the cost)**

Then use the Cut-and-Paste to probe optimal substructure

Given $i < j$

Suppose the optimal paranthesization of

$$A_i, A_{i+1}, ..., A_j$$

USE CONTRADICTION!

Now, the Recursion can be wrote!!!

Given that m[i,j] is the minimum number of scalar multiplications

$$m[i,j] = \begin{cases} 0 & \text{if } i == j \\ \min\limits_{i \leq k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

The Recursive Solution

Recursive Algorithm

1. Recursive-Matrix-Chain(p, i, j)
2. if $i == j$
3. return 0
4. $m[i, j] = \infty$
5. for $k = i$ to $j - 1$
6. $q =$ Recursive-Matrix-Chain(p, i, k) + ...
7. Recursive-Matrix-Chain$(p, k+1, j)$ +...
8. $p_{i-1}p_k p_j$
9. if $q < m[i, j]$
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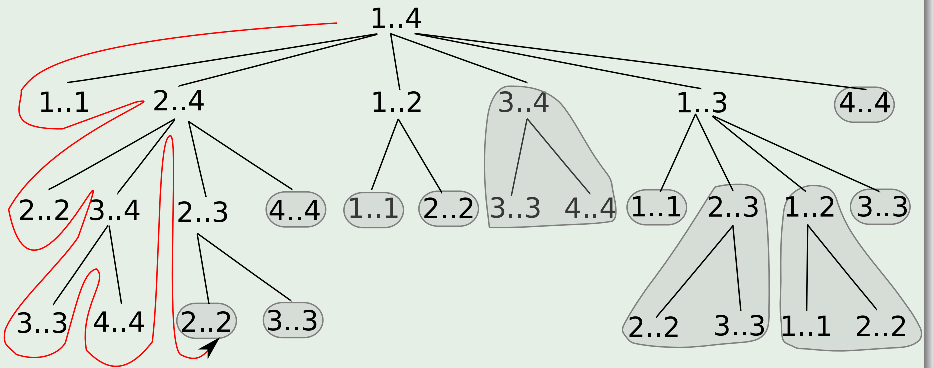
Again!!! Overlapping substructure

Red Line Represents the Recursion Path



This is a nightmare

We have the following recursion

$$\begin{aligned} T(1) &\geq 1, \\ T(n) &\geq 1 + \sum_{k=1}^{n-1} \left[T(n-k) + T(k) + 1\right] \text{ for } n > 1. \end{aligned}$$

First

Did you notice?

$T(i)$ appears once as $T(k)$ and once as $T(n-k)$ for $i = 1, 2, ..., n-1$.

We have then

$$T(n) \geq 1 + 2\sum_{i=1}^{n-1} [T(i)] + n - 1.$$

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Then

We decide to guess $T(n) = \Omega(2^n)$

- We shall guess the following $T(n) \geq 2^{n-1}$ for all $n \geq 1$
- First for $n = 1$ $T(1) \geq 1 = 2^0$

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Then

Now, for $n \geq 2$

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\begin{aligned}
T(n) &\geq 2\sum_{i=1}^{n-1}\left[T(i)\right]+n\\
&= 2\sum_{i=1}^{n-1}2^{i-1}+n\\
&= 2\sum_{i=0}^{n-2}2^{i}+n\\
&= 2\left(\frac{2^{n-1}-1}{2-1}\right)+n\\
&= 2\left(2^{n-1}-1\right)+n\\
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Thus

We want to avoid to calculate the same value many times

Use bottom up approach and store values at each step.

We get two arrays or tables

The first one, m

It is used to hold the information about the cost of multiplying the matrices

The second one, s

It is used to hold the place where the parenthesis is selected to minimize the cost

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How do we simulate the recursion Bottom-Up?

We do the following...

We use the following strategy:

- Solve the chain of matrices with small size (The smallest is 2 matrices... after all 1 matrix has cost 0)

Thus, we need

A loop from 2 to n for solving small sequences to larger ones.

In addition

An inner loop from 1 to $n-l+1$ (We do not want to get out of the sequence of matrices) for solving the smaller problems for the outer loop

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A value

j that is holding the ending index of the subsequence being taken in consideration.

Then a third loop

To go from i to $j-1$ to take the necessary decisions

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Bottom-Up Algorithm

MATRIX-CHAIN-ORDER(p)

1. **n = p.length-1**
2. **let** $m[1..n, 1..n]$ **and** $s[1..n-1, 2..n]$ **be new tables**
3. **for** $i = 1$ **to n**
4. $\quad m[i, i] = 0$
5. **for** $l = 2$ **to** n
6. $\quad$ **for** $i = 1$ **to** $n - l + 1$
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Example

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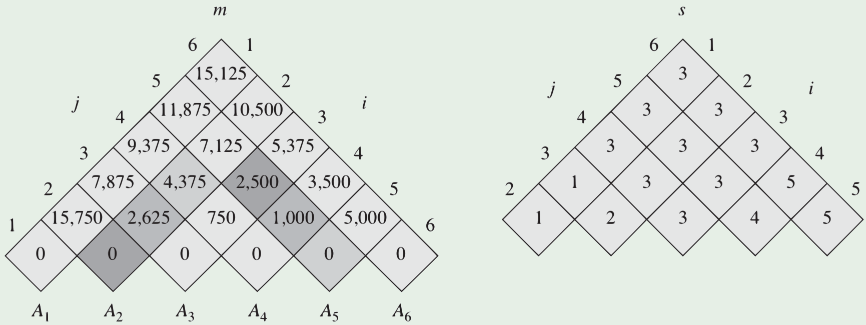
matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimensions	35×30	30×15	15×5	5×10	10×20	20×25



Complexity

By looking at the algorithm we have

$l \leftarrow n-1$

$i \leftarrow n-l-1$

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Then

$O(n^3)$

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Reconstruct the Output

PRINT-OPTIMAL-PARENS(s, i, j)

```
if i == j
        print "A_i"
else print "("
        PRINT-OPTIMAL-PARENS(s, i, s[i, j])
        PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
        print ")"
```

Final solution for the example

$((A_1(A_2A_3))((A_4A_5)A_6))$

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PRINT-OPTIMAL-PARENS(s, i, j)

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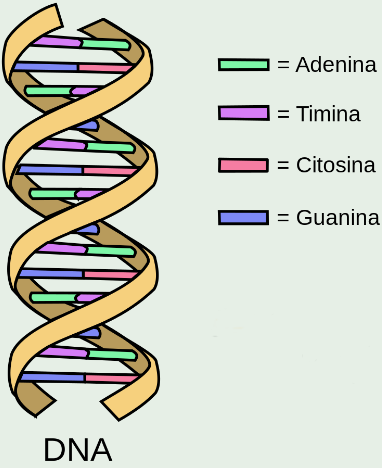
Outline

In Biology

Biological applications often need to compare the DNA of two (or more) different organisms.



Why?

Because given these strands

- S_1= ACCGGTCGAGTGCGCGGAAGCCGGCCGAA
- S_1= GTCGTTCGGAATGCCGTTGCTCTGTAAA

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- $S_1 =$ GTCGTTCGGAATGCCGTTGCTCTGTAAA

We want

To determine how "similar" the two strands are, as some measure of how closely related the two organisms are.

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For example

We can say that two DNA strands are similar if one is a substring of the other.

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This does not happen in the previous example...

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The Longer S_3

The more similar the organism, represented by S_1 and S_2, are.

Thus

We need to find S_3 the Longest Common Subsequence

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Longest Common Subsequence

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Given two sequences X and Y, we say that Z is a common subsequence of X and Y, if Z is a subsequence of both X and Y.

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Characterizing the LCS

Theorem 15.1 (Optimal substructure of an LCS)

Let $X = \langle x_1, x_2, ..., x_m \rangle$ and $Y = \langle y_1, y_2, ..., y_n \rangle$ be sequences, and let $Z = \langle z_1, z_2, ..., z_k \rangle$ be any LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
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Overlapping Property

To find an LCS for X and Y, we may need to find

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Thus

For the first case

$$Recursion(i, j) = Recursion(i - 1, j - 1) + 1$$

Second case

$$Recursion(i, j) = Recursion(i, j - 1)$$

However, you have the too

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Then, we can collapse second and third case

In the following way

$$Recursion(i,j) = \max\left\{Recursion(i-1,j), Recursion(i,j-1)\right\}$$

The Final Recurrence

Let $c[i,j]$ the length of the common subsequence of X_i, Y_j

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } i, j > \text{ and } x_i \neq y_j \end{cases}$$

Thus, we can do the following

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To develop an exponential algorithm.

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Why?

Clearly, we are using the bottom-up approach, so we get the results for the smallest problem first!!!

We also have a table to store the decisions

Ok, What type of symbols are in that table?

	y_i	a	v	c	r	e
x_i	0	0	0	0	0	0
a	0	$\nwarrow$				
b	0					
c	0			$\nwarrow$		
d	0					
e	0					$\nwarrow$

Thus, for the different cases

$x_m = y_n$

- Simply use the symbol " $\nwarrow$ ".
- After all we are consuming the same symbol

$c[i-1, j] \geq c[i, j-1]$

- Simply use the symbol " $\uparrow$ ".
- After all you are moving up in the rows

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How, we fill $c\left[0..m.0..n\right]$

Something Notable

We need to increase the columns and the rows.

Thus

- for $i = 1$ to m
-         for $j = 1$ to n

In addition, $c\left[0..m, 0\right]$ and $c\left[0, 0..n\right]$

If one of your subproblems is empty:

- We know that the common elements are 0.

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Final Algorithm - Complexity $O(mn)$

LCS-Length(X, Y)

1. $m = X.length$
2. $n = Y.length$
3. **let** $b[1..m, 1..n]$ **and** $c[0..m, 0..n]$ **be new tables**
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Example

The matrices after running the algorithm

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

Constructing the LCS

PRINT-LCS(b, X, i, j)

if $i == 0$ or $j == 0$

    return

if $b[i, j] ==$ "$\nwarrow$"

    PRINT-LCS$(b, X, i-1, j-1)$

    print x_i

elseif $b[i, j] ==$ "$\uparrow$"

    PRINT-LCS$(b, X, i-1, j)$

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Constructing the LCS

PRINT-LCS(b, X, i, j)

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if i == 0 or j == 0
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if b[i, j] == "↖"
        PRINT-LCS(b, X, i − 1, j − 1)
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elseif b[i, j] == "↑"
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Complexity

$O(m + n)$

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Exercises

From Cormen's book solve

- 15.3-3
- 15.3-5
- 15.2-3
- 15.2-4
- 15.2-5
- 15.4-2
- 15.4-4
- 15.4-5