Analysis of Algorithms

Binary Search Trees

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Outline

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Why Binary Search Trees?

Compared them with an array representation

Ouch!!! Insertion, Search and Deletion are quite expensive with the $O(n)$.

Instead Binary Search Trees

Since they are node based the cost of moving an element either into the collection or out of the collection is faster.

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Binary Search Tree Concepts

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A binary search tree (BST) is a data structure where each node posses three fields $left$, $right$ and p.

- They represent its left child, right child and parent.
- In addition, each node has the field key.

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Property

- Let x be a node in a binary search tree. If y is a node in the left subtree of x, then $key[y] \leq key[x]$.
- Similarly, if y is a node in the right subree of x, then $key[x] \leq key[y]$.

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In Order Walk

This walk allows to print the keys in sorted order!

Inorder-tree-walk(x)

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if $x \neq$ NIL
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Theorem 12.1

If x is the root of an n-node subtree, then the call Inorder-tree-walk(x) takes $\Theta(n)$ time.

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- Thus, you need to prove $T(n) = \Theta(n)$?

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Proof of inorder walk, $T(n) = O(n)$

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We have the following situation:

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Substitution Method

We have finally

$$T(n) = T(k) + T(n-k-1) + d$$

1. $T(k)$ is the amount of work done in the left
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3. $d > 0$ reflects an upper bound for the in-between work done for the print.

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We use the substitution method to prove that $T(n) = O(n)$

This can be done if we can bound $T(n)$ by bounding it by

$$(c+d)n + c \tag{1}$$

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This can be done if we can bound $T(n)$ by bounding it by

$$(c+d)n + c \tag{1}$$

Thus

For $n = 0$

$$T(0) = c = (c + d) \times 0 + c \tag{2}$$

Now, By Substitution Method

For $n > 0$

$$
\begin{aligned}
T(n) &\leq T(k) + T(n-k-1) + d \\
&= ((c+d)\,k + c) + ((c+d)\,(n-k-1) + c) + d \\
&= (c+d)\,n + c - (c+d) + c + d \\
&= (c+d)\,n + c
\end{aligned}
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Thus

$$T(n) = \Theta(n) \tag{3}$$

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$$T\left(n\right) = \Theta\left(n\right) \tag{3}$$

Outline

What may we use for a search?

Given a key k, we have the following Trichotomy Law

1. $x.key == k$
2. $x.key > k$
3. $x.key < k$

This allows us to take decisions

Go to the left or go to the right down the tree!!!

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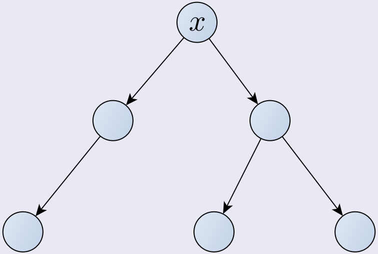
This allows us to take decisions

Go to the left or go to the right down the tree!!!

Case 1

Return Payload

if $(x.key == k)$ return $x.payload$



Searching

Searching

Tree-search(x, k)

```
if $x$ == NIL or $k$ == $x.key$
    return x
if $k < x.key$
    return Tree-search($x.left, k$)
else return Tree-search($x.right, k$)
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Complexity

$$O(h) \tag{4}$$

where h is the height of the tree ⇒ **we look for well balanced trees.**

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Minimum and Maximum

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Tree-minimum(x)

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while $x.left \neq$NIL
        $x = x.left$
return $x$
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1 while $x.left \neq$ NIL
2         $x = x.left$
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Tree-minimum(x)

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1. while $x.left$ ≠NIL
2.         $x = x.left$
3. return $x$
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where h is the height of the tree ⇒ **we look for well balanced trees.**

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Ouch!!!

At the End We Delete

- Thus, we have a problem!!!
- We need to maintain the Binary Search Property.

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A simple idea

Move the previous or next element to the deleted position!!!

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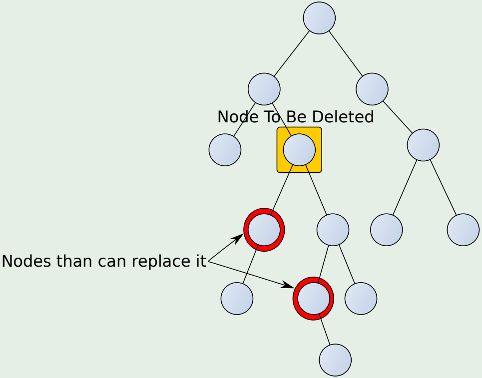
A simple idea

Move the previous or next element to the deleted position!!!

We want to do the following

We have then



Tree-Delete

TREE-DELETE(T, z)

```
if $z.left == $ NIL
    Transplant$(T, z, z.right)$
elseif $z.right == $ NIL
    Transplant$(T, z, z.left)$
else
    $y$=Tree-minimum$(z.right)$
    if $y.p \neq z$
        Transplant$(T, y, y.right)$
        $y.right = z.right$
        $y.right.p = y$
    Transplant$(T, z, y)$
    $y.left = z.left$
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```

Case 1

- Basically if the element z to be deleted has a NIL left child simply replace z with that child!!!

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    Transplant$(T, z, y)$
    $y.left = z.left$
    $y.left.p = y$
```

Case 2

- Basically if the element z to be deleted has a NIL right child simply replace z with that child!!!

Tree-Delete

TREE-DELETE(T, z)

```
if $z.left == $ NIL
    Transplant$(T, z, z.right)$
elseif $z.right == $ NIL
    Transplant$(T, z, z.left)$
else
    $y$=Tree-minimum$(z.right)$
    if $y.p \neq z$
        Transplant$(T, y, y.right)$
        $y.right = z.right$
        $y.right.p = y$
    Transplant$(T, z, y)$
    $y.left = z.left$
    $y.left.p = y$
```

Case 3

- The z element has not empty children you need to find the successor of it.

Tree-Delete

```
TREE-DELETE($T, z$)
if $z.left ==$ NIL
    Transplant($T, z, z.right$)
elseif $z.right ==$ NIL
    Transplant($T, z, z.left$)
else
    $y$=Tree-minimum($z.right$)
    if $y.p \neq z$
        Transplant($T, y, y.right$)
        $y.right = z.right$
        $y.right.p = y$
    Transplant($T, z, y$)
    $y.left = z.left$
    $y.left.p = y$
```

Case 4

- if $y.p \neq z$ then $y.right$ takes the position of y after all $y.left ==$ NIL
    - take $z.right$ and make it the new $right$ of y
    - make the $(y.right == z.right).p$ equal to y

Tree-Delete

TREE-DELETE(T, z)

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if $z.left == \text{NIL}$
    Transplant$(T, z, z.right)$
elseif $z.right == \text{NIL}$
    Transplant$(T, z, z.left)$
else
    $y$=Tree-minimum$(z.right)$
    if $y.p \neq z$
        Transplant$(T, y, y.right)$
        $y.right = z.right$
        $y.right.p = y$
    Transplant$(T, z, y)$
    $y.left = z.left$
    $y.left.p = y$
```

Case 4

- put y in the position of z
- make $y.left$ equal to $z.left$
- make the $(y.left == z.left).p$ equal to y

Support Operations: Transplant

Transplant(T, u, v)

```
if $u.p$ == NIL
    $T.root = v$
elseif $u == u.p.left$
    $u.p.left = v$
else $u.p.right = v$
if $v \neq$NIL
    $v.p = u.p$
```

Case 1

- If u is the root then make the root equal to v

Support Operations: Transplant

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if $v \neq$NIL
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```

Case 2

- if u is the left child make the left child of the parent of u equal to v

Support Operations: Transplant

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if $u.p == $ NIL
    $T.root = v$
elseif $u == u.p.left$
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else $u.p.right = v$
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```

Case 3

- Similar to the second case, but for right child

Support Operations: Transplant

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else $u.p.right = v$
if $v \neq$NIL
    $v.p = u.p$
```

Case 4

- If $v \neq$ NIL then make the parent of v the parent of u

Outline

1. Binary Search Trees Concepts
   - Introduction

2. Binary Search Tree Operations
   - Walking on a Tree
   - Searching
   - Minimum and Maximum
   - Deletion in Binary Search Trees
   - Examples of Deletion

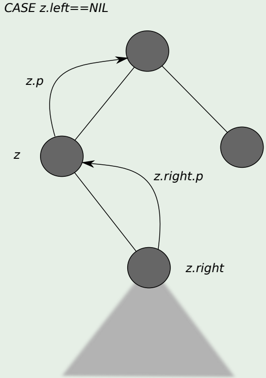
3. Balancing a Tree, AVL Trees
   - Adding a Height
   - The Height Problem
   - Insertions in AVL-Trees

4. Exercises
   - Some Excercises

Example: Deletion in BST

Case $z.left == NIL$

- if $z.left ==$ NIL
- Transplant$(T, z, z.right)$...



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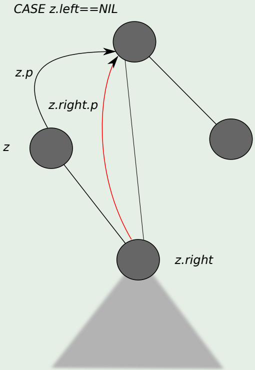


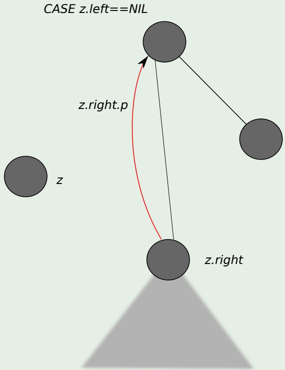
Transplant$(T, z, z.right)$

- elseif $z == z.p.left$
- $\quad z.p.left = z.right$
- if $z.right \neq$NIL
- $\quad z.right.p = z.p$

Example: Deletion in BST

Case $z.left == NIL$

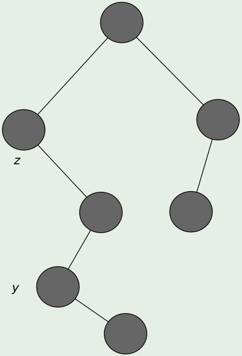
Remove the node z once you get out of the procedure



Another Example: Deletion in BST

Case $z.left \neq NIL$ and $z.right \neq NIL$

- y=Tree-minimum$(z.right)$



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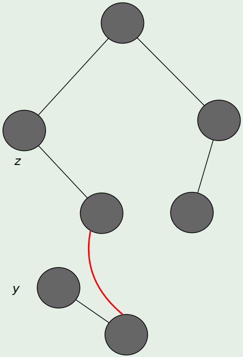
Case $z.left \neq NIL$ and $z.right \neq NIL$

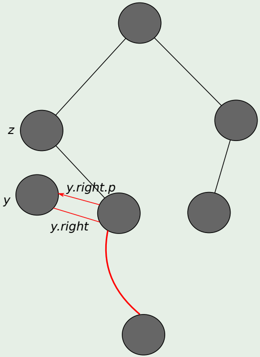
- if $y.p \neq z$
- Transplant$(T, y, y.right)$



Another Example: Deletion in BST

Case $z.left \neq NIL$ and $z.right \neq NIL$

- $y.right = z.right$
- $y.right.p = y$



Another Example: Deletion in BST

Case $z.left \neq NIL$ and $z.right \neq NIL$

- Transplant(T, z, y)
- $y.left = z.left$
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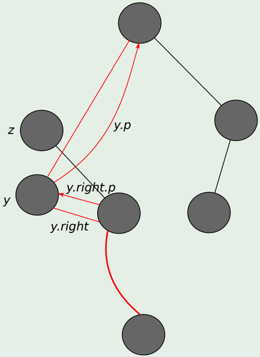

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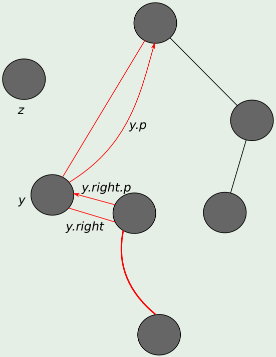

Outline

What do we need?

Tree Height

To describe AVL trees we need the concept of tree height

Definition

The maximal length of a path from the root to a leaf.

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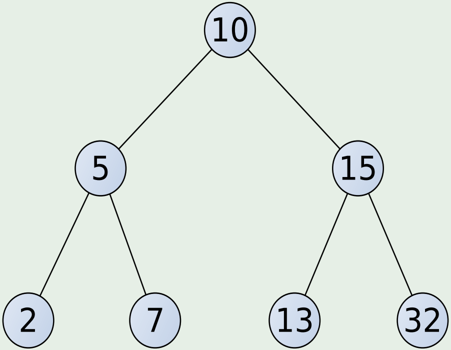
Example

Height = 3



Outline

We want the following

Height Invariant

At any node in the tree, the heights of the left and right sub-trees differs by at most 1.

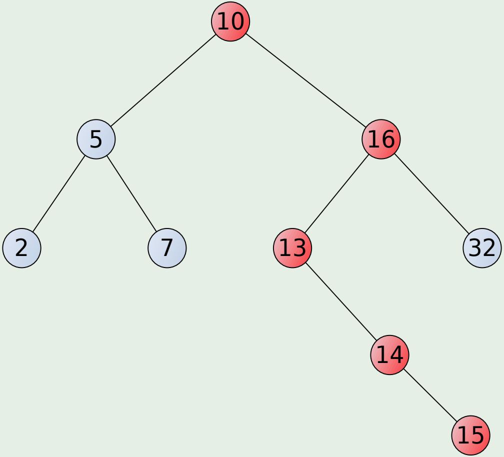
Thus, it is necessary to add an extra field to the Node Structure

The Code

```
class Node():
  def __init__():
    self.key = None
    self.height = 0
    self.Val = None
    self.left   = None
    self.right = None
```

Example

Violation of the Height Property



Outline

1 Binary Search Trees Concepts
- Introduction

2 Binary Search Tree Operations
- Walking on a Tree
- Searching
- Minimum and Maximum
- Deletion in Binary Search Trees
- Examples of Deletion

3 Balancing a Tree, AVL Trees
- Adding a Height
- The Height Problem
- Insertions in AVL-Trees

4 Exercises
- Some Excercises

Insertion

Similar to the Insertion in a BST

With a Fix-up at the end of the insertion

We have the following cases

1. Right Subtree is of height $h+1$ and the left subtree is of height h
2. Right Subtree is of height h and the left subtree is of height $h+1$

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- Now, we insert in the **right subtree** of the right subtree.
- The result of inserting into the **right subtree** will give us a new right subtree of height $h+2$.

This is how the tree looks like

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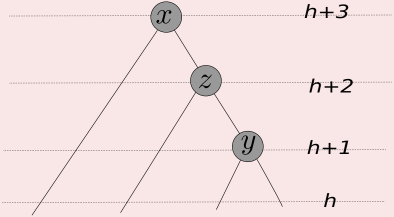

Then

This

Which raises the height of the overall tree to $h+3$

In addition

In the new right subtree has height $h+2$

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We can do the following

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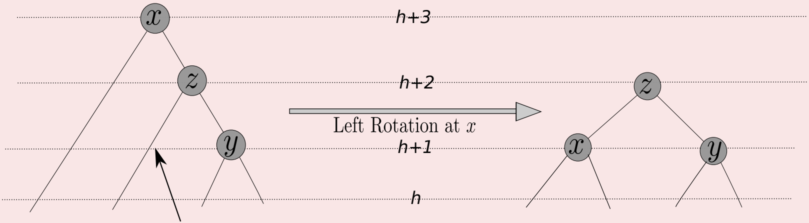

There is no left node at this level

Now, The second case

We insert into the right subtree

But now the left subtree of the right subtree has height $h + 1$.

Example

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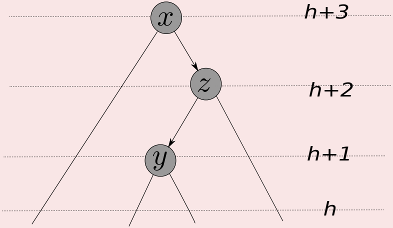
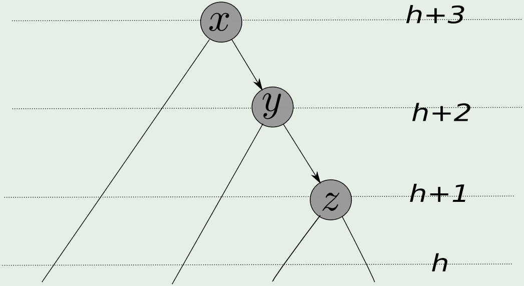


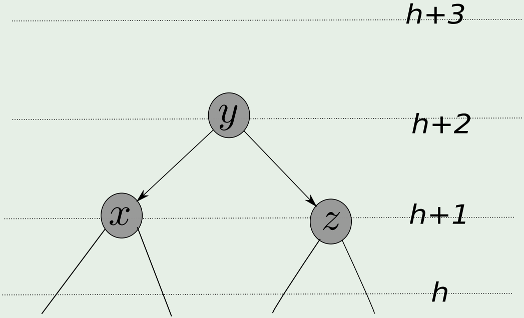
We fix the problem by

First a right rotation with respect to the z



We fix the problem by

Now a left rotation with respect to the x



Outline

Excercises

From Cormen's book, chapters 11 and 12

- 11.1-2
- 11.2-1
- 11.2-2
- 11.2-3
- 11.3-1
- 11.3-3
- 12.1-3
- 12.1-5
- 12.2-5
- 12.2-7
- 12.2-9
- 12.3-3