Analysis of Algorithms

Hash Tables

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About Basic Data Structures

Remark

It is quite interesting to notice that many data structures actually share similar operations!!!

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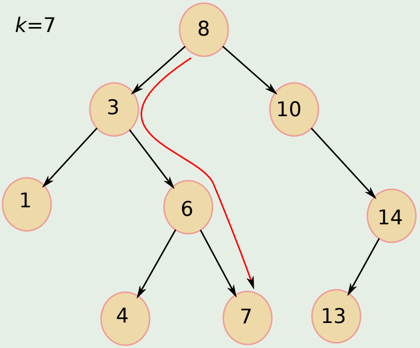
Examples

Search(S,k)

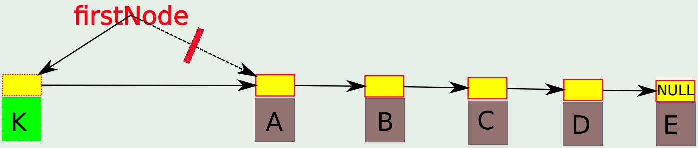
Example: Search in a BST



Examples

Insert(S,x)

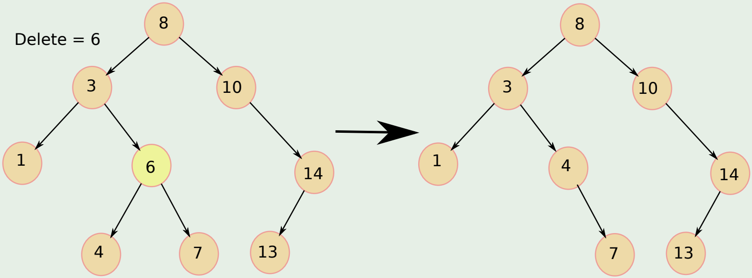
Example: Insert in a linked list



And Again

Delete(S,x)

Example: Delete in a BST



Basic data structures and operations.

Therefore

This are basic structures, it is up to you to read about them.

- Chapter 10 Cormen's book

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Hash tables: Concepts

Definition

- A hash table or hash map T is a data structure, most commonly an array, that uses a hash function to efficiently map certain identifiers of keys (e.g. person names) to associated values.

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- Key values are direct addresses in the array.
- Direct implementation or Direct-address tables.

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   - return $T[k]$
2. Direct-Address-Search(T, x)
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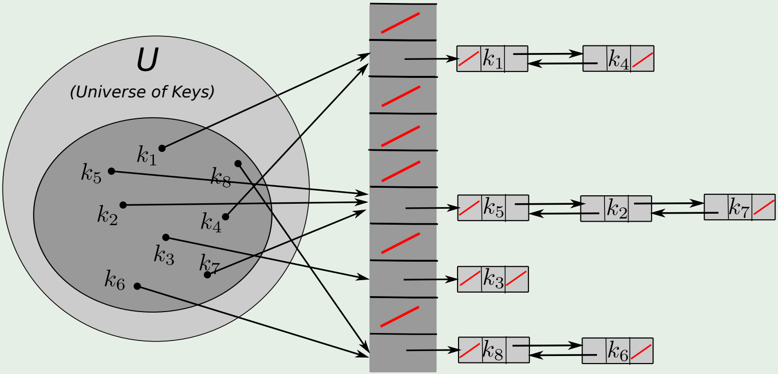
Hash tables: Chaining

A Possible Solution

Insert the elements that hash to the same slot into a linked list.



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Analysis of hashing with Chaining: Assumptions

Assumptions

- We have a load factor $\alpha = \frac{n}{m}$, where m is the size of the hash table T, and n is the number of elements to store.
- **Simple uniform hashing** property:
  - This means that any of the m slots can be selected.
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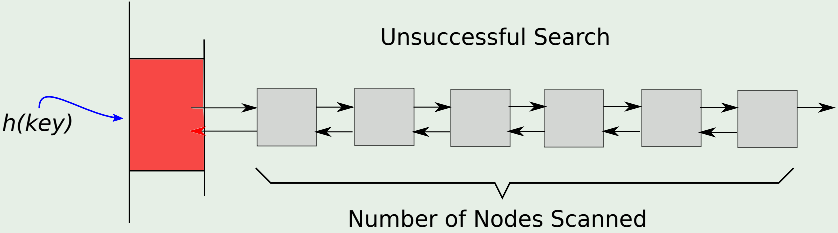

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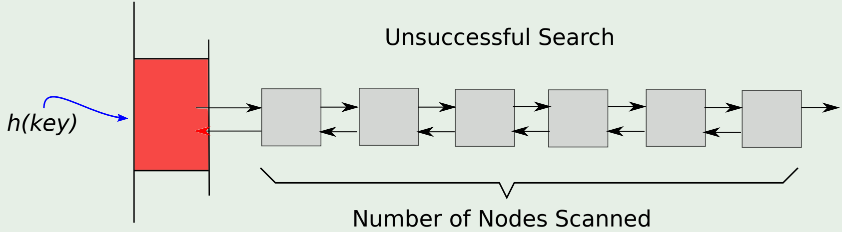

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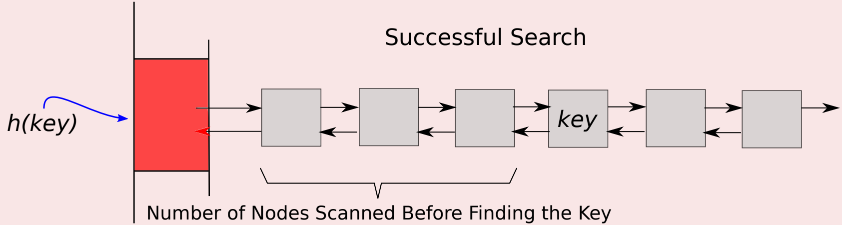

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In a hash table in which collisions are resolved by chaining, an unsuccessful search takes average-case time $\Theta(1+\alpha)$, under the assumption of simple uniform hashing.

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Analysis of hashing: Constant time.

Finally

These two theorems tell us that if $n = O(m)$

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Analysis of hashing: Which hash function?

Consider that:

Good hash functions should maintain the property of simple uniform hashing!

- The keys have the same probability $1/m$ to be hashed to any bucket!!!
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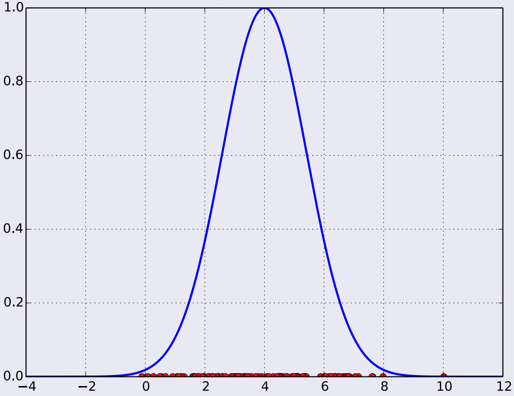
What if...

Question:

What about something with keys in a normal distribution?



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The division method

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Keys interpreted as natural numbers

Given a string "pt", we can say p = 112 and t=116 (ASCII numbers)

- ASCII has 128 possible symbols.
  - Then $(128 \times 112) + 128^0 \times 116 = 14452$

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- $m = 2^p$, $h(k)$ is only the p lowest-order bits.
- $m = 2^p - 1$, when k is interpreted as a character string interpreted in radix 2^p, permuting characters in k does not change the value.

- For example, given $n = 2000$ elements
  - We can use $m = 701$ because it is near to $2000/3$ but not near a power of two.

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Second

Then, select an s in the range $0 < s < 2^w$ and assume $A = \frac{s}{2^w}$.

Third

Now, we multiply k by the number $s = A2^w$.

Example

Fourth

The result of that is $r_1 2^w + r_0$, a $2w$-bit value word, where the first p-most significative bits of r_0 are the desired hash value.

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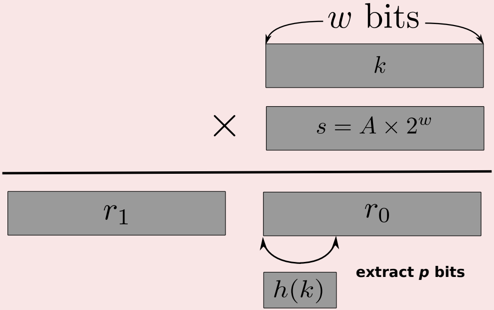

Outline

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Sooner or Latter

We can pick up a hash function that does not give us the desired uniform randomized property

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We are required to analyze the possible clustering of the data by the hash function

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Hash table do not give a way to measure clustering

Thus, table designers

They should provide some clustering estimation as part of the interface.

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The clustering measure needs an estimate of the variance of the distribution of bucket sizes.

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Measuring Clustering through a metric C

Definition

If bucket i contains n_i elements, then

$$C = \frac{m}{n-1}\left[\frac{\sum_{i=1}^{m} n_i^2}{n} - 1\right] \tag{2}$$

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Properties

1. If $C = 1$, then you have uniform hashing.
2. If $C > 1$, it means that the performance of the hash table is slowed down by clustering by approximately a factor of C.
3. If $C < 1$, the spread of the elements is more even than uniform!!! Not going to happen!!!

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If clustering is occurring, some buckets will have more elements than they should, and some will have fewer.

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There will be a **wider range of bucket sizes** than one would expect from a random hash function.

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Consider the following random variable

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Then, given

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We look at the Variance of X_{ij}

We look at the dispersion of X_{ij}

$$Var\left[X_{ij}\right]=E\left[X_{ij}^{2}\right]-\left(E\left[X_{ij}\right]\right)^{2}=\frac{1}{m}-\frac{1}{m^{2}} \tag{5}$$

What about the expected number of elements at each bucket?

$$E\left[n_i\right]=E\left[\sum_{j=1}^{n}X_{ij}\right]=\frac{n}{m}=\alpha \tag{6}$$

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Then, we have given independence of $\{X_{ij}\}$

Because independence of $\{X_{ij}\}$, the scattering of n_i

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We can plug back on C using the expected value

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Using a hash table that enforce a uniform distribution in the buckets

- We get that $C = 1$ or the best distribution of keys

Outline

Now, we have a really horrible hash function $\equiv$ It hits only one of every b buckets

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$$E\left[X_{ij}\right] = E\left[X_{ij}^2\right] = \frac{b}{m} \tag{11}$$

Thus, we have

$$E\left[n_i\right] = \alpha b \tag{12}$$

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A Possible Solution, Universal Hashing

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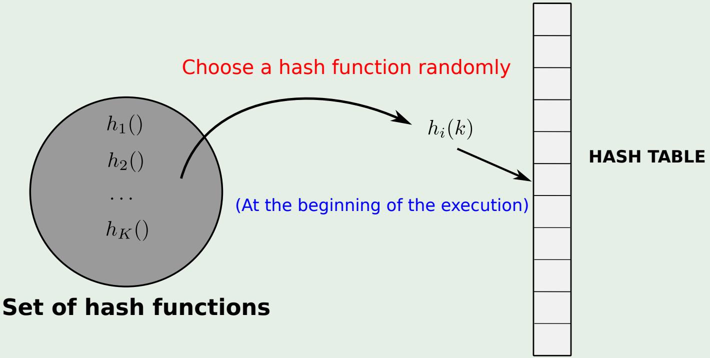
Universal hashing

Example



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Definition of Universal Hash Functions

Definition

Let $H = \{h : U \to \{0, 1, ..., m-1\}\}$ be a family of hash functions. H is called a universal family if

$$\forall x, y \in U, x \neq y : \Pr_{h \in H}(h(x) = h(y)) \leq \frac{1}{m} \tag{14}$$

Main result

With universal hashing the chance of collision between distinct keys k and l is no more than the $\frac{1}{m}$ chance of collision if locations $h(k)$ and $h(l)$ were randomly and independently chosen from the set $\{0, 1, ..., m-1\}$.

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Universal Hashing

Theorem 11.3

- Suppose that a hash function h is chosen randomly from a universal collection of hash functions and has been used to hash n keys into a table T of size m, using chaining to resolve collisions.
- If key k is not in the table, then the expected length $E[n_{h(k)}]$ of the list that key k hashes to is at most the load factor $\alpha = \frac{n}{m}$. If key k is in the table, then the expected length $E[n_{h(k)}]$ of the list containing key k is at most $1 + \alpha$.

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Using universal hashing and collision resolution by chaining in an initially empty table with m slots, it takes expected time $\Theta(n)$ to handle any sequence of n INSERT, SEARCH, and DELETE operations $O(m)$ INSERT operations.

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Example of Universal Hash

Proceed as follows:

- Choose a primer number p large enough so that every possible key k is in the range $[0, ..., p-1]$

$$\mathbb{Z}_p = \{0, 1, ..., p-1\} \text{and } \mathbb{Z}_p^* = \{1, ..., p-1\}$$

- Define the following hash function:

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Outline

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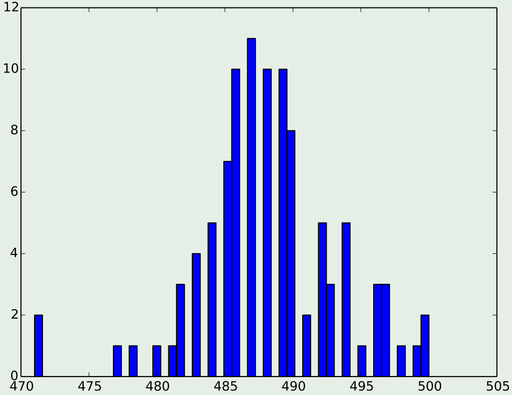
- $p = 977$, $m = 50$, a and b random numbers
  - $h_{a,b}(k) = ((ak + b) \mod p) \mod m$

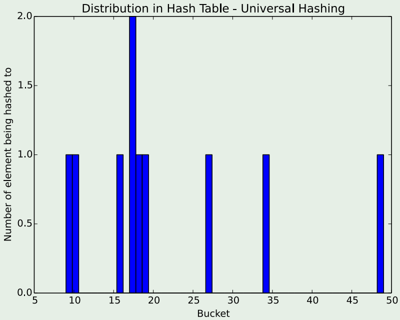
Example of key distribution

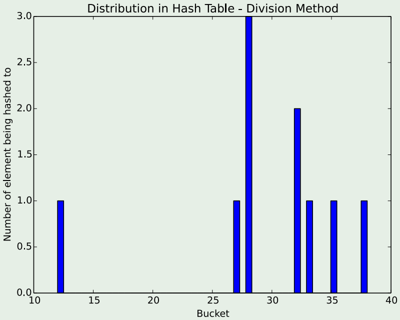
Example, mean = 488.5 and dispersion = 5



Example with 10 keys

Universal Hashing Vs Division Method





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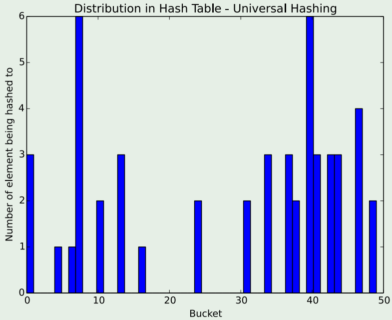

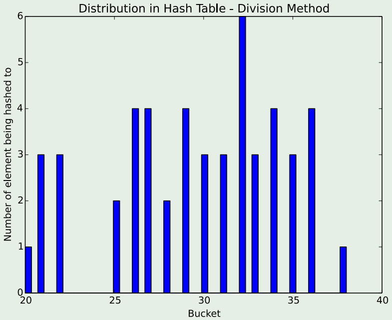

Example with 100 keys

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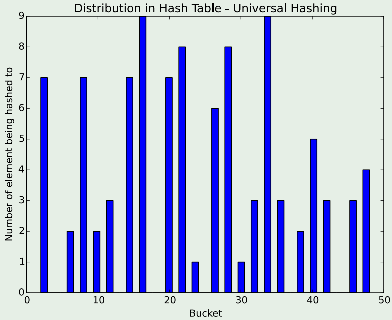
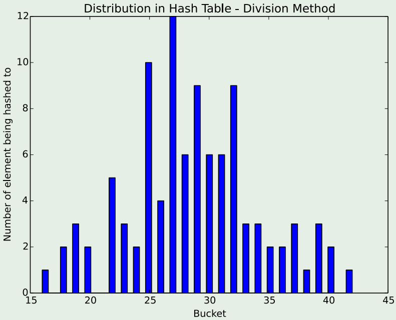
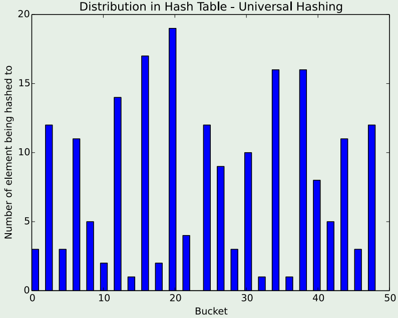
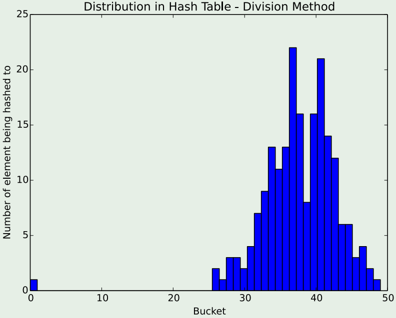




Example with 200 keys

An example of $P(\Theta|X) = P(X|\Theta) P(\Theta)$





Another Example: Matrix Method

Then

- Let us say keys are u-bits long.
- Say the table size M is power of 2.
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 & u
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Thus

- The column i does not contribute to the final answer of $h(l)$ because of the zero!!!

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Implementation of the column*vector mod 2

Code

```
int product(int row, int vector){

  int i = row & vector;

  i = i - ((i >> 1) & 0x55555555);
  i = (i & 0x33333333) + ((i >> 2) & 0x33333333);
  i = (((i + (i >> 4)) & 0x0F0F0F0F) * 0x01010101) >> 24;

  return i & i & 0x00000001;

}
```

Advantages of universal hashing

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- Universal hashing provides good results on average, independently of the keys to be stored.
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Definition

All the elements occupy the hash table itself.

What is it?

We systematically examine table slots until either we find the desired element or we have ascertained that the element is not in the table.

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Extended hash function to **probe**

- Instead of being fixed in the order $0, 1, 2, ..., m-1$ with $\Theta(n)$ search time.
- Extend the hash function to $h : U \times \{0, 1, ..., m-1\} \to \{0, 1, ..., m-1\}$.
- This gives the probe sequence $\langle h(k, 0), h(k, 1), ..., h(k, m-1) \rangle$.
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Hashing methods in Open Addressing

HASH-INSERT(T, k)

1. $i = 0$
2. **repeat**
3. $\quad j = h(k, i)$
4. $\quad$ **if** $T[j] == NIL$
5. $\quad\quad T[j] = k$
6. $\quad\quad$ **return** j
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8. **until** $i == m$
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until $T[j] == NIL$ or $i == m$
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Hashing methods in Open Addressing

HASH-SEARCH(T,k)

$i = 0$

repeat

  $j = h(k, i)$

  if $T[j] == k$

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Outline

Linear probing: Definition and properties

Hash function

- Given an ordinary hash function $h' : U \rightarrow \{0, 1, ..., m-1\}$ for $i = 0, 1, ..., m-1$, we get the extended hash function

$$h(k, i) = (h'(k) + i) \mod m, \tag{15}$$

Sequence of probes

Given key k, we first probe $T[h'(k)]$, then $T[h'(k)+1]$ and so on until $T[m-1]$. Then, we wrap around $T[0]$ to $T[h'(k)-1]$.

Distinct probes

Because the initial probe determines the entire probe sequence, there are m distinct probe sequences.

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Linear probing: Definition and properties

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- Linear probing suffers of primary clustering.
- Long runs of occupied slots build up increasing the average search time.
- Long runs of occupied slots tend to get longer, and the average search time increases.

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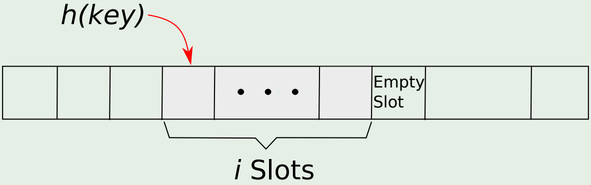
Clusters arise because an empty slot preceded by i full slots gets filled next with probability $\frac{i+1}{m}$.



Thus

The probability of getting a collision increases dramatically after each insertion.

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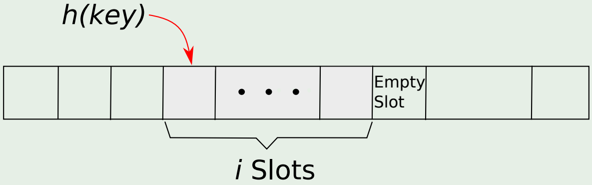

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Example

Example using keys uniformly distributed

It was generated using the division method

Then

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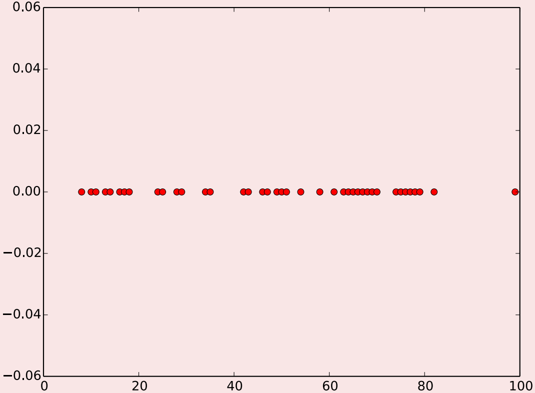

Example

Example using Gaussian keys

It was generated using the division method

Then

Example

Example using Gaussian keys

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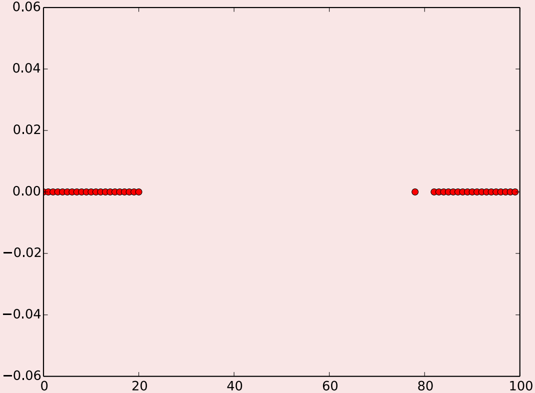

Outline

Linear Probing, Insertion and Deletion

Constraints

- Divisor $= m$ (number of buckets) $= 17$.
- Home bucket $=$ key % 17.

Then

Put in pairs whose keys are 6, 12, 34, 29, 28, 11, 23, 7, 0, 33, 30, 45

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We have

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

Outline

Linear Probing – Remove

Example

0				4				8				12				16
34	0	45				6	23	7			28	12	29	11	30	33

remove(0)

Compact Cluster

Search cluster for pair (if any) to fill vacated bucket.

Linear Probing – Remove

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
34	0	45				6	23	7			28	12	29	11	30	33

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Linear Probing – remove(34)

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Linear Probing – remove(29)

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34	0					6	23	7			28	12	11	30	45	33

Code for Removing

We have the following

```
public void remove(key){
    int positionsChecked = 1;
    int i = FindSlot(Key);
    if (Table[i] == null)
        return;   // key is not in the table
    j = i;
    while(positionsChecked <= Table.length){
        j = (j+1) % Table.length;
        if (Table[j] == null) break;
        k = Hashing(Table[j].key);
        if (i < j && (k <= i || k > j)) ||
           (j < i && (k <= i && k > j)) {
                Table[i] = Table[j];
                i = j;
        }
        positionChecked++;
    }
    Table[i] = null;
}
```

Explanation

First

For all records in a cluster, there must be no vacant slots between their natural hash position and their current position (else lookups will terminate before finding the record).

Second

- k is the raw hash where the record at j would naturally land in the hash table if there were no collisions.

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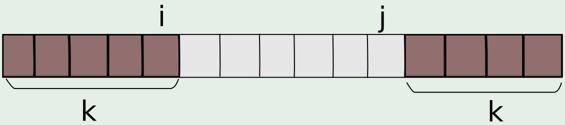

we have $i < j$

- If $i < k \leq j$ then moving j to the i position will be incorrect... Why?

Case 2

We have the following

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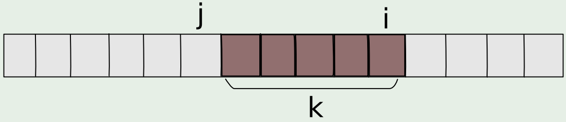

We have $j < i$

- If $k \leq j <$ or $i < k$ then moving j to the i position will be incorrect... Why?

Outline

Quadratic probing: Definition and properties

Hash function

- Given an auxiliary hash function $h' : U \rightarrow \{0, 1, ..., m-1\}$ for $i = 0, 1, ..., m-1$, we get the extended hash function

$$h(k, i) = (h'(k) + c_1 i + c_2 i^2) \mod m, \tag{16}$$

where c_1, c_2 are auxiliary constants

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- Given key k, we first probe $T[h'(k)]$, later positions probed are offset by amounts that depend in a quadratic manner on the probe number i.
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Advantages

This method works much better than linear probing, but to make full use of the hash table, the values of c_1,c_2, and m are constrained.

Disadvantages

If two keys have the same initial probe position, then their probe sequences are the same, since $h(k_1, 0) = h(k_2, 0)$ implies $h(k_1, i) = h(k_2, i)$. This property leads to a milder form of clustering, called secondary clustering.

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Definition and properties

Hash function

Double hashing uses a hash function of the form

$$h(k,i) = (h_1(k) + ih_2(k)) \mod m, \qquad (17)$$

where $i = 0, 1, ..., m-1$ and h_1, h_2 are auxiliary hash functions (Normally for a Universal family)

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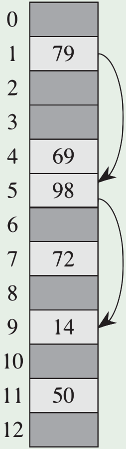
Example

Jumping around to insert 14 with $h_1(k) = k \mod 13$ and $h_2(k) = 1 + (k \mod 11)$



Outline

Analysis of Open Addressing

Theorem 11.6

Given an open-address hash table with load factor $\alpha = \frac{n}{m} < 1$, the expected number of probes in an unsuccessful search is at most $\frac{1}{1-\alpha}$ assuming uniform hashing.

Corollary

Inserting an element into an open-address hash table with load factor requires at most $\frac{1}{1-\alpha}$ probes on average, assuming uniform hashing.

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Given an open-address hash table with load factor $\alpha < 1$, the expected number of probes in a successful search is at most $\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$ assuming uniform hashing and assuming that each key in the table is equally likely to be searched for.

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Exercise's

From Cormen's book, chapters 11

- 11.1-2
- 11.2-1
- 11.2-2
- 11.2-3
- 11.3-1
- 11.3-3