Analysis of Algorithms

Medians and Order Statistics

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Outline

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Introduction

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The ith order statistic of a set of n elements is the ith smallest element.

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Input:

A set A of n (distinct) numbers and an integer i, with $1 \leq i \leq n$.

Output:

The element $x \in A$ that is larger than exactly $i-1$ other elements of A.

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Minimum-Maximum

Minimum using $n-1$ comparissons

Minimum(A)

1. $min = A[1]$
2. **for** $i = 2$ **to A.length**
3.   **if** $min > A[i]$
4. $min = A[i]$
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- Minimum: The first element.
- Maximum: The last element.
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- For the median:
  - If n is odd, then the median is equal to the $\frac{n+1}{2}$ th element.
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All selections can be done in $O(1)$, so total: $O(n \log n)$.

Question!!!

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$3\lfloor \frac{n}{2} \rfloor$comparisons.

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Selection in expected linear time $O(n)$

Selecting in expected linear time implies:

- Selecting the ith element.
- Use the divide and conquer algorithm RANDOMIZED-SELECT.
  - Similar to Quicksort, partition the input array recursively.
  - Unlike Quicksort, which works on both sides of the partition, just work on one side of the partition. This is called PRUNE-AND-SEARCH, prune one side, just search the other.

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- How? Remember Randomized Quicksort!!!

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We get a q from the random partition!!!

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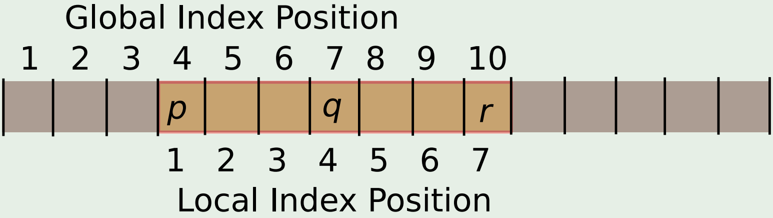
Example

Positions



p-q=3 then k=p-q+1 =4

Thus, the Strategy

If $i < k$

The possible ith smallest element is between p and $q - 1$.

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If $i > k$

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RANDOMIZED-SELECT

RANDOMIZED-SELECT Algorithm

Randomized-Select(A, p, r, i)

1. **if** $p == r$
2. **return** $A[p]$
3. $q =$ Randomized-Partition(A, p, r)
4. $k = q - p + 1$ // **Local Index**
5. **if** $i == k$ // **The Answer**
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Analysis of RANDOMIZED-SELECT

Worst-case running time $\Theta(n^2)$. Why?

An empty side and a side with remaining elements. So every partitioning of m elements will take $\Theta(m)$ time where $m = n, n-1, ..., 2$. Thus in total

$$\Theta(n) + \Theta(n-1) + ... + \Theta(2) = \Theta(\frac{n(n-1)}{2} - 1) = \Theta(n^2).$$

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Select the ith smallest element of $S = \{a_1, a_2, ..., a_n\}$.

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Solution:

- Use the so called PRUNE-AND-SEARCH technique:
  - Let $x \in S$, and partition S into three subsets.
  - $S_1 = \{a_j | a_j < x\}$, $S_2 = \{a_j | a_j = x\}$, $S_3 = \{a_j | a_j > x\}$.
  - If $|S_1| > i$, search ith smallest elements in S_1 recursively, (prune S_2 and S_3 away).
  - Else If $|S_1| + |S_2| > i$, then return x (the ith smallest element).
  - Else search the $(i - (|S_1| + |S_2|))$th element in S_3 recursively (prune S_1 and S_2 away).

A question arises

How to select x such that S_1 and S_3 are nearly equal in cardinality? Force an even search!!!

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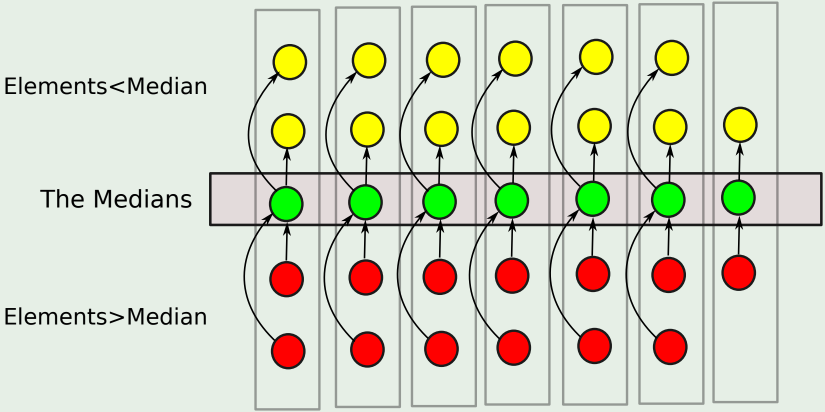

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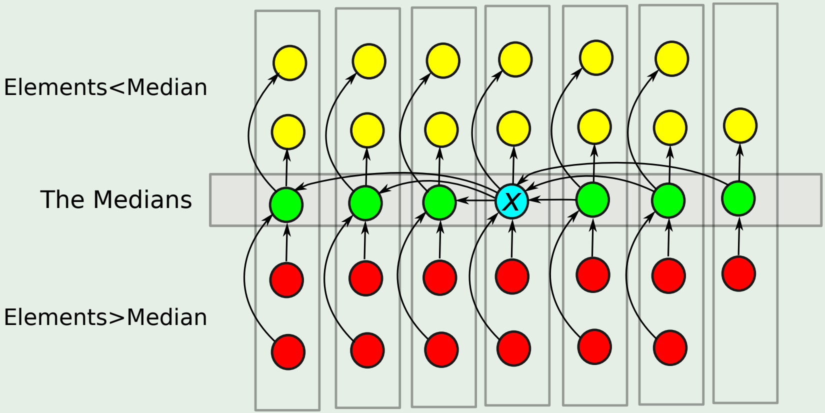

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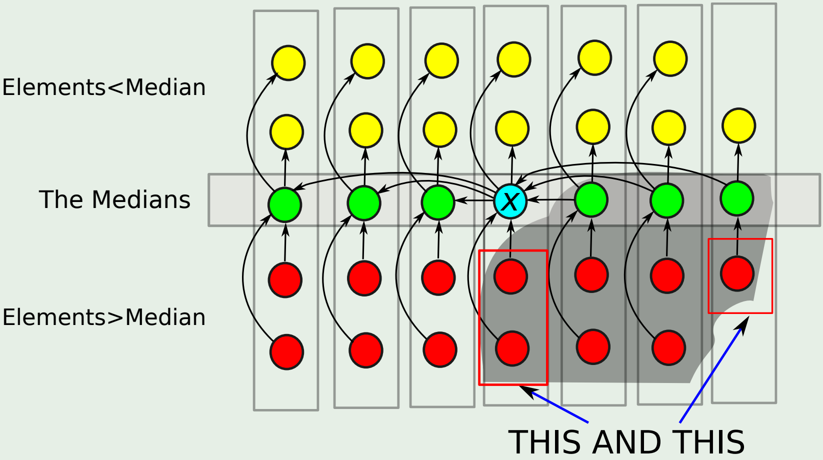

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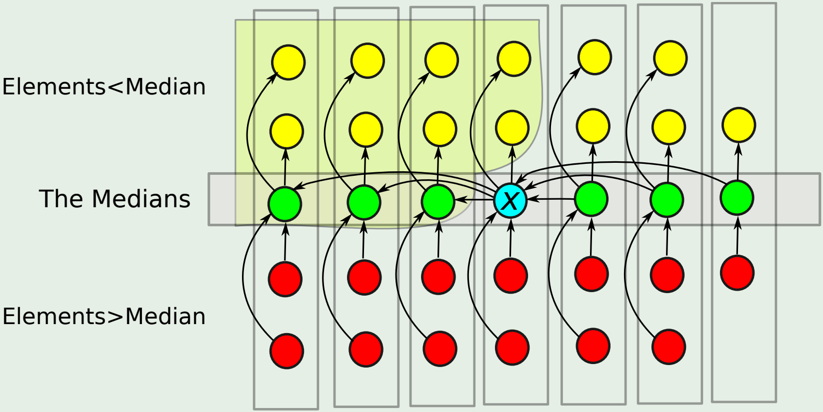

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2. Find the median of each group.
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4. Partition n elements around x into S_1, S_2, and S_3.
5. If $|S_1| > i$, search ith smallest element in S_1 recursively.
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- Similarly, the number of elements less or equal than x is also at least $\frac{3n}{10} - 6$.
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Final Recursion

We have then

$$T(n) = \begin{cases} O(1) & \text{if } n < \text{some value (i.e. 140)} \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + O(n) & \text{if } n \geq \text{some value (i.e. 140)} \end{cases}$$

Solve recurrence by substitution

Suppose $T(n) \leq cn$ for some c

$$T(n) \leq c\lceil \frac{n}{5} \rceil + c\left(\frac{7n}{10} + 6\right) + an$$

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- Given a neighborhood of n members of x, you need to find the median to substitute the value in x

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Exercises

From Cormen's Book Chapter 9

- 9.1-1
- 9.2-3
- 9.3-4
- 9.3-8
- 9.2