Analysis of Algorithms

The Mathematics of Analysis of Algorithms

Andres Mendez-Vazquez

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Outline

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Basic Induction

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer.

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Then the statement

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We have the following method for Mathematical Induction

Consider a statement of the form

$$\text{For all integers } n \geq a,\ P(n) \text{ is true.} \tag{2}$$

To prove such a statement

Perform the following two steps

Step 1 (Basis step)

Show that $P(a)$ is true - we normally use $a = 1$.

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Step 2 (Inductive step)

Show that for all integers $k \geq a$, if $P(k)$ is true, then $P(k+1)$ is true.

Inductive hypothesis

Suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

Then, you can prove that

$P(k+1)$ is true.

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Example

Proposition

For all integers $n \geq 8$, $n¢$ can be obtained using $3¢$ and $5¢$ coins.

Show that P(8) is true

$P(8)$ is true because $8¢$ can be obtained using one coin $3¢$ and another coin of $5¢$.

Show that for all integers $k \geq 8$, if $P(k)$ is true then $P(k+1)$ is also true

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Inductive hypothesis

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Suppose that k is any integer with $k \geq 8$ such that

- $k¢$ can be obtained using $3¢$ and $5¢$ coins.

Case 1 - There is a $5¢$ among those making the change for $k¢$

In this case, replace $5¢$ by two $3¢$. Thus, we get the change for $(k+1)¢$

- At least three $3¢$ coins must have been used
- Remove those coins and replaced them using two $5¢$
- The result will be $(k+1)¢$

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Case 2 - There is not a 5¢ among those making the change for k¢

- Because $k \geq 8$, at leat three coins must have been used.
- At least three 3¢ coins must have been used.
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- The result will be $(k+1)\,¢$.

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We can go further...

Recursively defined sets and structures

Assume S is a set. We use two steps to define the elements of S.

Basis Step

Specify an initial collection of elements.

Recursive Step

Give a rule for forming new elements from those already known to be in S.

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Thus

Let S be a set that has been defined recursively

To prove that every object in S satisfies a certain property:

1. Show that each object in the BASE for S satisfies the property.
2. Show that for each rule in the RECURSION, if the rule is applied to objects in S that satisfy the property, then the objects defined by the rule also satisfy the property.

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Example: Binary trees recursive definition

Recall that the set B of binary trees over an alphabet A is defined as follows

1. Basis: $\langle\rangle \in B$
2. Recursive definition: If $L, R \in B$ and $x \in A$ then $< L, x, R >\in B$.

Now define the function $f : B \to \mathbb{N}$ defined as

$$f(\langle\rangle) = 0$$

$$f(\langle L, x, R\rangle) = \begin{cases} 1 & \text{if } L = R = \langle\rangle \\ f(L) + f(R) & \text{otherwise} \end{cases}$$

Theorem

Let T in B be a binary tree. Then $f(T)$ yields the number of leaves of T.

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Let T in B be a binary tree. Then $f\left(T\right)$ yields the number of leaves of T.

Proof

By structural induction on T

Basis: The empty tree has no leaves, so $f(\langle\rangle) = 0$ is correct.

Induction

Let L, R be trees in B, $x \in A$.

Now

Suppose that $f(L)$ and $f(R)$ denotes the number of leaves of L and R, respectively.

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Case 1

If $L = R = \langle\rangle$, then $\langle L, x, R\rangle = \langle\langle\rangle, x, \langle\rangle\rangle$ has one leaf, namely x, so $f(\langle\langle\rangle, x, \langle\rangle\rangle) = 1$ is correct.

Case 2

If L and R are both not empty, then the number of leaves of the tree $\langle L, x, R\rangle$ is equal to the number of leaves of L plus the number of leaves of R.

$$f(\langle L, x, R\rangle) = f(L) + f(R) \tag{3}$$

Q.E.D.

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Introduction

When an algorithm contains an iterative control construct such as a while or for loop

It is possible to express its running time as a series:

$$\sum_{j=1}^{n} j \tag{4}$$

Thus, what is the objective of using these series

To be able to find bounds for the complexities of algorithms

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Definition of Series

Definition

Given a sequence $a_1, a_2, ..., a_n$ of numbers, where n is a no-negative integer, we can say that

$$a_1 + a_2 + ... + a_n = \sum_{k=1}^{n} a_k \tag{5}$$

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In the case of infinite series

$$a_1 + a_2 + ... = \sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k \tag{6}$$

Here, we have concepts of convergence and divergence that I will allow you to study.

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Outline

Linearity

For any real number c and any finite sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$

$$\sum_{k=1}^{n} [ca_k + b_k] = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \tag{7}$$

For More

Please take a look at page 1146 of Cormen's book.

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Telescopic Sum

Observation

- In certain sums each term is a difference of two quantities.
- You sometimes see that all the terms cancel except the first and the last.

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Example

Imagine that each term in the sum has the following structure:

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)} \tag{8}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} \tag{9}$$

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Definition

For any sequence $a_0, a_1, ..., a_n$,

$$\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0 \tag{10}$$

Similarly

$$\sum_{k=0}^{n-1} (a_k - a_{k+1}) = a_0 - a_n \tag{11}$$

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Arithmetic series

Summing over the set $\{1, 2, 3, ..., n\}$

We can prove that

$$\sum_{i=1}^{n} i = \frac{n\left(n+1\right)}{2} \tag{12}$$

Proof

Basis. If $n = 1$ then $\frac{1\times 2}{2} = 1$

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Proof

Induction, assume that is true for n

$$\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n + 1$$

$$= \frac{n(n+1)}{2} + n + 1$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

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Series of Squares and Sums

Series of Squares

$$\sum_{k=0}^{n} k^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6} \quad (13)$$

Series of Cubes

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Geometric Series

Definition

For a real $x \neq 1$, we have that

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + ... + x^n \tag{15}$$

It is called the geometric series

It is possible to prove that

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} \tag{16}$$

Proof

Now multiply both sides by of (Eq. 15) by x

$$x \left[\sum_{k=0}^{n} x^k \right] = x + x^2 + x^3 + ... + x^{n+1} \tag{17}$$

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Proof

Subtract (Eq. 17) from (Eq. 15)

$$\sum_{k=0}^{n} x^k - x\left[\sum_{k=0}^{n} x^k\right] = 1 - x^{n+1} \tag{18}$$

Finally

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x} = \frac{x^{n+1} - 1}{x - 1} \tag{19}$$

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Subtract (Eq. 17) from (Eq. 15)

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Infinite Geometric Series

When the summation is infinite and $|x| < 1$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{20}$$

Proof

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$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{20}$$

Proof

Given that

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For more on the series

Please take a look to

Cormen's book - Appendix A.

Outline

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This is quite useful for Analysis of Algorithms

Important

The most basic way to evaluate a series is to use mathematical induction.

Example

Prove that

$$\sum_{k=0}^{n} 3^k \leq c3^n \qquad (22)$$

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Fast Bounding of Series

A quick upper bound on the arithmetic series

$$\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} n = n^2 \tag{23}$$

In general, for a series $\sum_{k=1}^{n} a_k$

If $a_{\max} = \max_{1 \leq k \leq n} a_k$ then

$$\sum_{k=1}^{n} a_k \leq n \cdot a_{\max} \tag{24}$$

Another fast way of bounding finite series is

Suppose that $\frac{a_{k+1}}{a_k} \leq r$ for all $k \geq 0$ where $0 < r < 1$.

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Approximation by integrals

When a summation has the from $\sum_{k=m}^{n} f(k)$, where $f(k)$ is a monotonically increasing function

$$\int_{m-1}^{n} f(x)\, dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)\, dx \quad (26)$$

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For example

Given

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}dx \leq \sum_{k=1}^{n} \frac{1}{k} \tag{27}$$

In addition

$$\sum_{k=1}^{n} \frac{1}{k} \leq \int_1^{n} \frac{1}{x}dx = \ln n \tag{28}$$

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Outline

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Gerolamo Cardano: Gambling out of Darkness

Gambling

Gambling shows our interest in quantifying the ideas of probability for millennia, but exact mathematical descriptions arose much later.

Gerolamo Cardano (16th century)

While gambling he developed the following rule!!!

Equal conditions

“The most fundamental principle of all in gambling is simply equal conditions, e.g. of opponents, of bystanders, of money, of situation, of the dice box and of the dice itself. To the extent to which you depart from that equity, if it is in your opponent’s favour, you are a fool, and if in your own, you are unjust.”

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Gerolamo Cardano's Definition

Probability

"If therefore, someone should say, I want an ace, a deuce, or a trey, you know that there are 27 favourable throws, and since the circuit is 36, the rest of the throws in which these points will not turn up will be 9; the odds will therefore be 3 to 1."

Meaning

Probability as a ratio of favorable to all possible outcomes!!! As long all events are equiprobable...

Thus, we get

$$P(\text{All favourable throws}) = \frac{\text{Number All favourable throws}}{\text{Number of All throws}} \tag{29}$$

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Empiric Definition

Intuitively, the probability of an event A could be defined as:

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Where $N(A)$ is the number that event a happens in n trials.

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Outline

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Axioms of Probability

Axioms

Given a sample space S of events, we have that

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. If $A_1, A_2, ..., A_n$ are mutually exclusive events (i.e. $P(A_i \cap A_j) = 0$), then:

$$P(A_1 \cup A_2 \cup ... \cup A_n) = \sum_{i=1}^{n} P(A_i)$$

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Set Operations

We are using

Set Notation

Thus

What Operations?

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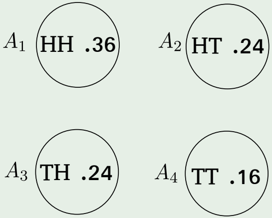
Example

Setup

Throw a biased coin twice



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At least one head!!! Can you tell me which events are part of it?

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Tail on first toss.

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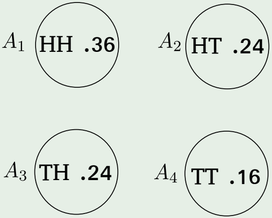

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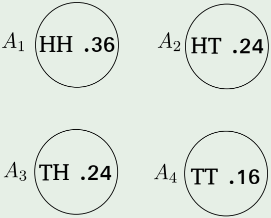

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We have four main methods of counting

1. Ordered samples of size r with replacement
2. Ordered samples of size r without replacement
3. Unordered samples of size r without replacement
4. Unordered samples of size r with replacement

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Ordered samples of size r with replacement

Definition

The number of possible sequences $(a_{i_1}, ..., a_{i_r})$ for n different numbers is

$$n \times n \times ... \times n = n^r \quad (30)$$

Example

If you throw three dices you have $6 \times 6 \times 6 = 216$

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$$n \times n-1 \times ... \times (n-(r-1)) = \frac{n!}{(n-r)!} \tag{31}$$

Example

The number of different numbers that can be formed if no digit can be repeated. For example, if you have 4 digits and you want numbers of size 3.

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Actually, we want the number of possible unordered sets.

However

We have $\frac{n!}{(n-r)!}$ collections where we care about the order. Thus

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Use a digit trick for that

Look at the Board

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How?

Change encoding by adding more signs

Imagine all the strings of three numbers with $\{1, 2, 3\}$

We have

Old String	New String
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112	1+0,1+1,2+2=124
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Outline

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Independence

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Two events A and B are independent if and only if
$P(A, B) = P(A \cap B) = P(A)P(B)$

Do you have any example?

Any idea?

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We can use it to derive the Binomial Distribution

WHAT?????

First, we use a sequence of n Bernoulli Trials

We have this

- "Success" has a probability p.
- "Failure" has a probability $1 - p$.

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Thus, taking a sample ω

$\omega = 11 \cdots 10 \cdots 0$

k 1's followed by $n - k$ 0's.

We have then

$$
\begin{aligned}
P(\omega) &= P\left(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}^c \cap \ldots \cap A_n^c\right) \\
&= P(A_1)\, P(A_2) \cdots P(A_k)\, P\left(A_{k+1}^c\right) \cdots P\left(A_n^c\right) \\
&= p^k (1-p)^{n-k}
\end{aligned}
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Important

The number of such sample is the number of sets with k elements.... or...

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Did you notice?

We do not care where the 1's and 0's are

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Thus, we are looking to sum all those probabilities of all those combinations of 1's and 0's

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Proving this is a probability

Sum of these probabilities is equal to 1

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) p\,(1-p)^{n-k} = (p + (1-p))^{n} = 1$$

The other is simple

$$0 \leq \left(\begin{array}{c} n \\ k \end{array} \right) p\,(1-p)^{n-k} \leq 1 \;\forall k$$

This is know as

The Binomial probability function!!!

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Outline

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Different Probabilities

Unconditional

This is the probability of an event A prior to arrival of any evidence, it is denoted by $P(A)$. For example:

- P(Cavity)=0.1 means that "in the absence of any other information, there is a 10% chance that the patient is having a cavity".

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Posterior Probabilities

Relation between conditional and unconditional probabilities

- Conditional probabilities can be defined in terms of unconditional probabilities:

$$P(A|B) = \frac{P(A, B)}{P(B)}$$

which generalizes to the **chain rule** $P(A, B) = P(B)P(A|B) = P(A)P(B|A)$.

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Law of Total Probabilities

- if $B_1, B_2, ..., B_n$ is a partition of mutually exclusive events and A is an event, then $P(A) = \sum_{i=1}^{n} P(A \cap B_i)$. An special case $P(A) = P(A,B) + P(A,\overline{B})$.
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Example

Three cards are drawn from a deck

Find the probability of no obtaining a heart

We have

- 52 cards
- 39 of them not a heart

Define

A_i ={Card i is not a heart} Then?

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Independence and Conditional

From here, we have that...

$P(A|B) = P(A)$ and $P(B|A) = P(B)$.

Conditional independence

A and B are conditionally independent given C if and only if

$$P(A|B, C) = P(A|C)$$

Example: $P(WetGrass|Season, Rain) = P(WetGrass|Rain)$.

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Bayes Theorem

One Version

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- $P(A|B)$ is the conditional probability of A, given B. It is also called the posterior probability because it is derived from or depends upon the specified value of B.
- $P(B|A)$ is the conditional probability of B given A. It is also called the likelihood.
- $P(B)$ is the prior or marginal probability of B, and acts as a normalizing constant.

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General Form of the Bayes Rule

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Example

Setup

Throw two unbiased dice independently.

Let

1. A ={sum of the faces =8}
2. B ={faces are equal}

Then calculate $P(B|A)$

Look at the board

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We have the following

Two coins are available, one unbiased and the other two headed

Assume

That you have a probability of $\frac{3}{4}$ to choose the unbiased

- $B_1 = \{\text{Unbiased coin chosen}\}$
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  - Find that if a head come up, find the probability that the two headed coin was chosen

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Random Variables

Definition

In many experiments, it is easier to deal with a summary variable than with the original probability structure.

- Suppose we record a "1" for agree and "0" for disagree.
- The sample space for this experiment has 2^{50} elements. Why?
- Suppose we are only interested in the number of people who agree.
- Define the variable X = number of "1"'s recorded out of 50.
- Easier to deal with this sample space (has only 51 elements).

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It is necessary to define a function random variable as follow

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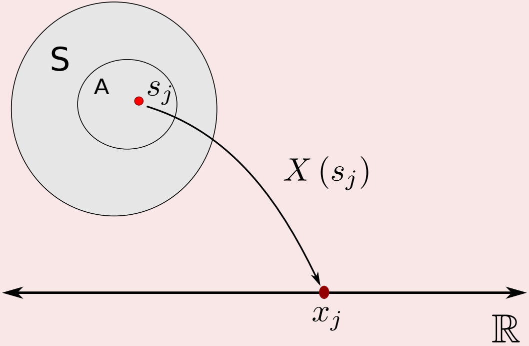

Random Variables

How?

What is the probability function of the random variable is being defined from the probability function of the original sample space?

- Suppose the sample space is $S = \{s_1, s_2, ..., s_n\}$
- Suppose the range of the random variable $X =< x_1, x_2, ..., x_m >$
- Then, we observe $X = x_i$ if and only if the outcome of the random experiment is an $s_j \in S$ s.t. $X(s_j) = x_j$ or

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Setup

Throw a coin 10 times, and let R be the number of heads.

Then

S = all sequences of length 10 with components H and T

We have for

$\omega =$HHHHTTHTTH $\Rightarrow R(\omega) = 6$

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Types of Random Variables

Discrete

A discrete random variable can assume only a countable number of values.

Continuous

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Probability Mass Function (PMF) and Probability Density Function (PDF)

The pmf /pdf of a random variable X assigns a probability for each possible value of X.

- $\sum_x p(x) = 1$ and $P(a < X < b) = \sum_{k=a}^{b} p(k)$
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Outline

Cumulative Distributive Function I

Cumulative Distribution Function

- With every random variable, we associate a function called Cumulative Distribution Function (CDF) which is defined as follows:

$$F_X(x) = P(f(X) \leq x)$$

- With properties:
  - $F_X(x) \geq 0$
  - $F_X(x)$ in a non-decreasing function of X.

Example

- If X is discrete, its CDF can be computed as follows:

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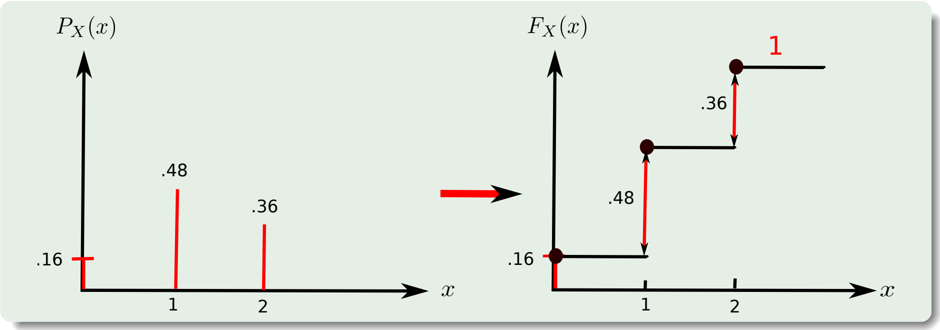
Example: Discrete Function



Cumulative Distributive Function II

Continuous Function

If X is continuous, its CDF can be computed as follows:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

Remark

Based in the fundamental theorem of calculus, we have the following equality.

$$p(x) = \frac{dF}{dx}(x)$$

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This particular $p(x)$ is known as the Probability Mass Function (PMF) or Probability Distribution Function (PDF).

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Example: Continuous Function

Setup

- A number X is chosen at random between a and b
- X has a uniform distribution
  - $f_X(x) = \frac{1}{b-a}$ for $a \leq x \leq b$
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- A number X is chosen at random between a and b
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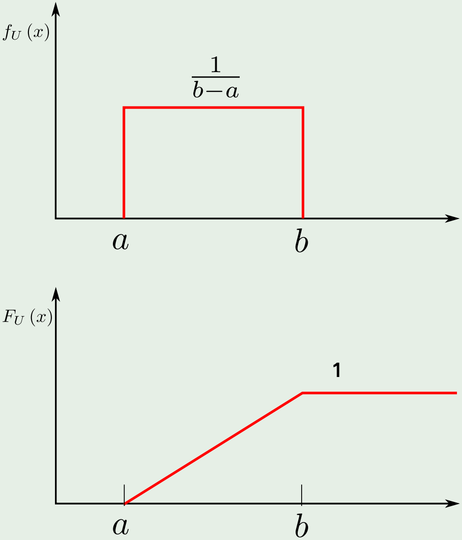
Graphically

Example uniform distribution



Outline

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Properties of the PMF/PDF

Conditional PMF/PDF

We have the conditional pdf:

$$p(y|x) = \frac{p(x,y)}{p(x)}.$$

From this, we have the general chain rule

$$p(x_1, x_2, ..., x_n) = p(x_1|x_2, ..., x_n)p(x_2|x_3, ..., x_n)...p(x_n).$$

Independence

If X and Y are independent, then:

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Expectation

Something Notable

You have the random variables R_1, R_2 representing how long is a call and how much you pay for an international call:

$$\text{if } 0 \leq R_1 \leq 3(minute) \;\; R_2 = 10(cents)$$

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$P\{R_2 = 10\} = 0.6$, $P\{R_2 = 20\} = 0.25$, $P\{R_2 = 10\} = 0.15$.

We can say that we have $N \times 0.6$ calls and $10 \times N \times 0.6$ is the cost of those calls

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Linear Property

Linearity property of the Expected Value

$$E(af(X) + bg(Y)) = aE(f(X)) + bE(g(Y)) \quad (36)$$

Example for a discrete distribution

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Imagine the following

We have the following functions

1. $f(x) = e^{-x}, \; x \geq 0$
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Definition

- $Var(X) = E((X-\mu)^2)$ where $\mu = E(X)$

Standard Deviation

- The standard deviation is simply $\sigma = \sqrt{Var(X)}$.

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Outline

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Definition

The indicator of an event A is a random variable I_A defined as follows:

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \quad (37)$$

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Method of Indicators

It is possible to see random variables as a sum of indicators functions

$$R = I_{A_1} + ... + I_{A_n} \tag{39}$$

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Example

A single unbiased die is tossed independently n times

- Let R_1 be the numbers of 1's.
- Let R_2 be the numbers of 2's.

Find $E[R_1 R_2]$

We can express each variable as

$$R_1 = I_{A_1} + ... + I_{A_n}$$
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Case 2 $i = j$ A_i and B_i are disjoint

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