Analysis of Algorithms

Introduction

Andres Mendez-Vazquez

September 2, 2018

Outline

Outline

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Informal definition

Informally, an algorithm is any well defined computational procedure that

- It takes some value, or set of values, as input.
- Then, it produces some value, or set of values, as output.

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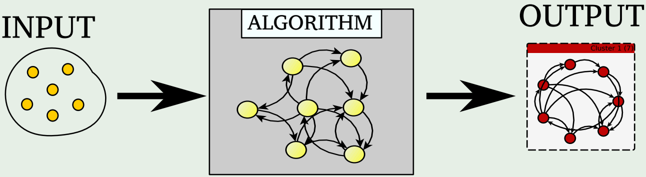

Example

Sorting Problem

- **Input:** A sequence of N numbers $a_1, a_2, ..., a_N$
- **Output:** A reordering of the input sequence $a_{(1)}, a_{(2)}, ..., a_{(N)}$

Actually

We are dealing with **instances of a problem.**

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Stuff Like

A sequence of integer numbers

10	2	4	5	11	36	18	9	50

The Classic

- We want to order the numbers!!!

Stuff Like

A sequence of integer numbers

10	2	4	5	11	36	18	9	50

The Classic

- We want to order the numbers!!!

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Instance of a Problem

Instance of the problem

- For example, we have
  - 9, 8, 5, 6, 7, 4, 3, 2, 1
- Then, we finish with
  - 1, 2, 3, 4, 5, 6, 7, 8, 9

Although Instances are Important

Nevertheless

The way we use those instances is way more important

For example

Look at Recursive Fibonacci!!!

Although Instances are Important

Nevertheless

The way we use those instances is way more important

For example

Look at Recursive Fibonacci!!!

Example: Fibonacci

Fibonacci rule

- $F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n > 1 \\ 1 & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$

Time Complexity

1. Naive version using directly the recursion - exponential time.
2. A more elegant version - linear time.

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Kolmogov's Definition

A bound for each sub-step

- An algorithmic process splits into steps whose complexity is bounded in advance
  - i.e., the bound is independent of the input and the current state of the computation.

Ending the Process

- The process runs until either the next step is impossible or a signal says the solution has been reached.

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Transformations done at each step

- Each step consists of a direct and immediate transformation of the current state.
- This transformation applies only to the active part of the state and does not alter the remainder of the state.

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By The Way (BTW)

How do they look this machines, this algorithms?

After all we like to see them!!!

An Example of an Algorithm

Insertion Sort

Data: Unsorted Sequence A

Result: Sort Sequence A

`Insertion Sort(A)`

for $j \leftarrow 2$ **to** *lenght(A)* **do**

  $key \leftarrow A[j]$;

  `// Insert` $A[j]$ `Insert` $A[j]$ `into the sorted sequence` $A[1,...,j-1]$

  $i \leftarrow j-1$;

  while $i > 0$ *and* $A[i] > key$ **do**

    $A[i+1] \leftarrow A[i]$;

    $i \leftarrow i-1$;

  end

  $A[i+1] \leftarrow key$

end

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Single-Source Shortest Path

Application → Short Paths in Maps

These algorithms allows to solve the problem of finding the shortest path in a map between two addresses.

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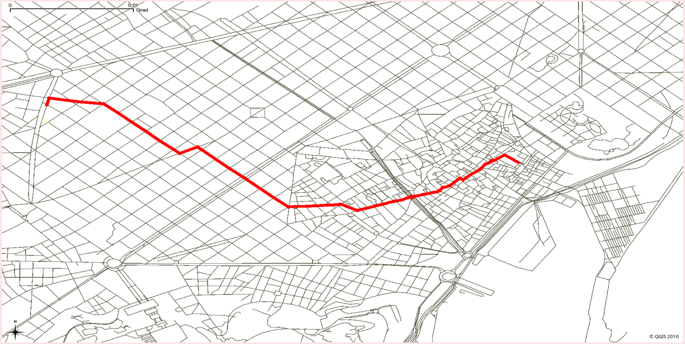

Solving Systems of Linear Equations

Application → Inverting Matrices

Because of stability reasons, given the system $Ax = y$, we use the the LUP decomposition or Cholensky decomposition to obtain the inverse A^{-1}.

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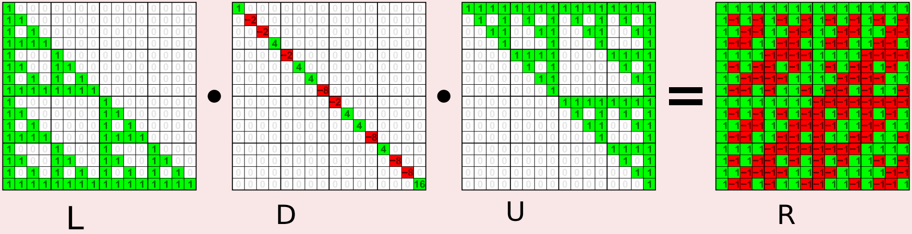

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Huffman Codes

Application → Compression

This method is part of the greedy methods. They are used for compression, they can achieve 20% to 90% compression in text files.

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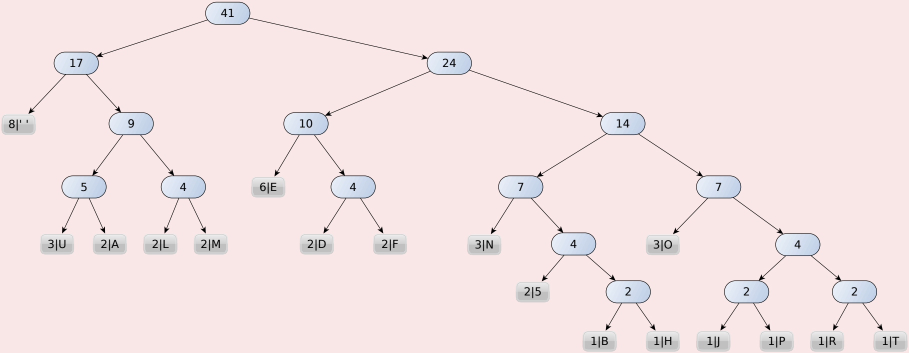

Matrix Multiplication

Application → Fast Multiplication of Matrices

In many algorithms, we want to multiply different $n \times n$ matrices.

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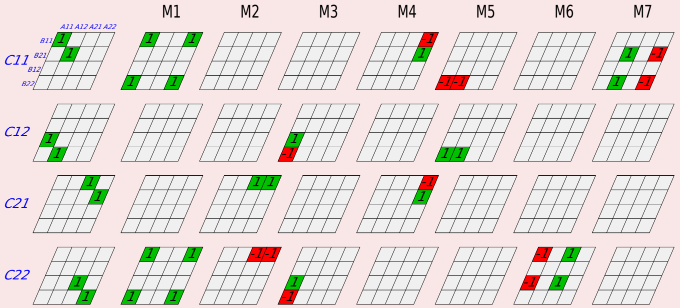

STRASSEN'S ALGORITHM

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Convex Hull

Application → Computational Geometry

Given the points in a plane, we want to find the minimum convex hull that encloses them.

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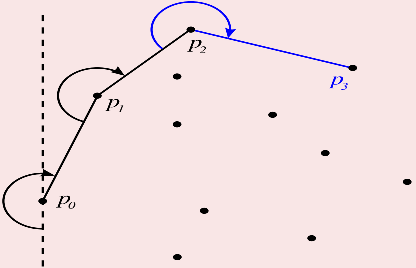

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Application → Computational Molecular Engineering

- In this field the engineers and biologist try to use the basis of life to create complex molecular machines.
- All these machines will requiere complex algorithms.

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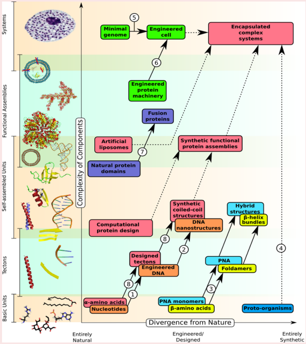

Pattern Recognition

Application → Machine Learning

In Machine Learning, we try to find specific patterns in data.

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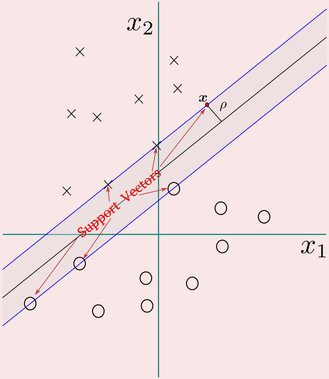

Databases

Application → Partition of the Database Space

For fast access Queries!!!

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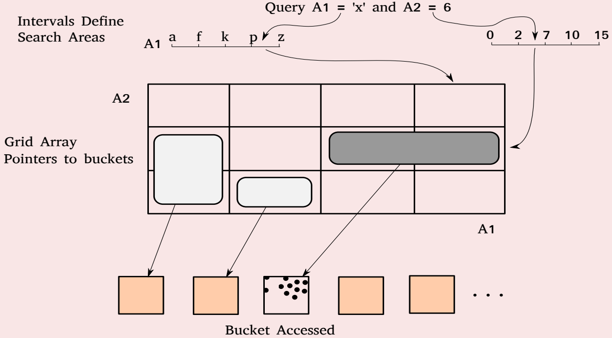

@Copyright Michael Unwalla: A mixed transaction cost model for coarse grained multi-column partitioning in a shared-nothing database machine

Databases

Application → Face Recognition

Facial Recognition measure face landmarks to identify different features in the face.

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Growth Functions

- Asymptotic Notation - Ω, O and Θ
- Standard notation and common functions
- Solving Recursions

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- The substitution method
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- Encodings
- Polynomial Time Verification
- Polynomial Reduction
- NP-Hard
- NP-Complete proofs
- A family of NP-Problems

- Branch and Bound

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Now, what we are going to look at!!!

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Some stuff about notation!!!

Second

What abstraction of a Computer to use?

Third

A first approach to analyzing algorithms!!!

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Insertion Sort(A)

for $j \leftarrow 2$ to length(A)
    do
        $key \leftarrow A[j]$
        ▶Insert $A[j]$ into the sorted sequence $A[1, ..., j-1]\}$
        $i \leftarrow j - 1$
        while $i > 0$ and $A[i] > key$
            do
                $A[i+1] \leftarrow A[i]$
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Rule

- Always put the name of the algorithm at the top together with the input.

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Rule

- Always initialize all the variables.
- The $a \leftarrow b$(You also can use "$=$.") means that the value b is passed to a.

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Rule

- Use indentation to preserve the block structure avoiding clutter.

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```

Rule

- ▶ it corresponds to comments and you can also use "//"

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The Random-Access Machine

Definition

A Random Access Machine (RAM) is an abstract computational-machine model identical to a multiple-register counter machine with the addition of indirect addressing.

- It contains arithmetic instructions found in low level languages
- It has control instructions: Conditional and unconditional branches, return and call functions.
- It is able to do data movement: load, store, copy.
- It posses data types: integer and floating point.

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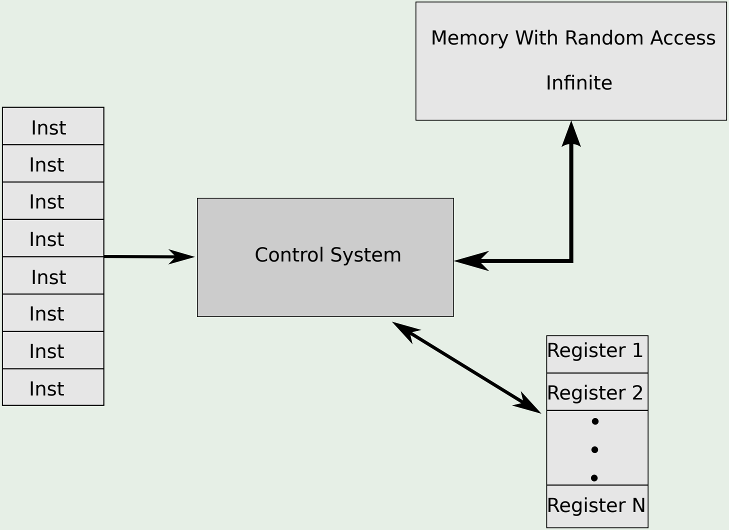
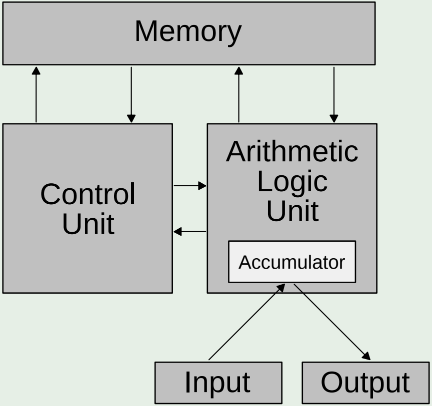
RAM Model

We have that



Although there are other equivalent models

Von Neumann architecture scheme



Outline

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Input Size and Running Time

Definition

The Input Size depends on the type of problem. We will indicate which input size is used per problem.

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The Running Time of an algorithm is the number of of primitives operations or steps executed. For now, we will assume that each line in an algorithm takes c_i a constant time.

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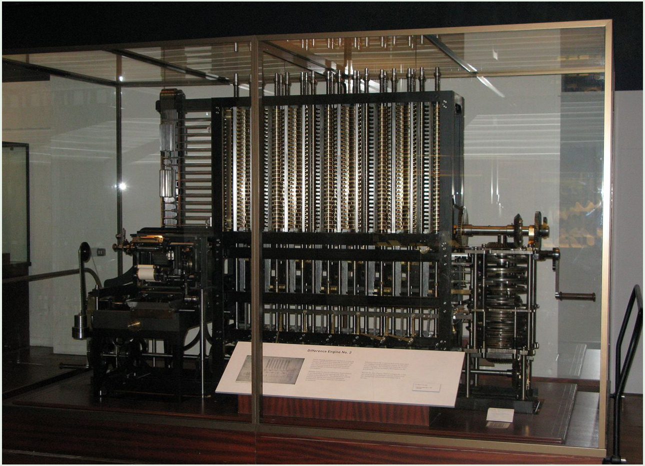
Even Babbage cared about how many turns of the crank were necessary!!!

Look at the crank!!!



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We are going to do some quite simple

Counting the number of operations

- Therefore we have the following equivalences using algebraic sums...

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Loops equivalent to Sums

$$\text{while } i > 0 \text{ and } A[i] > key \longleftrightarrow \sum_{j=2}^{N} 1 = N - 1$$

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Counting the Operations

Insertion Sort(A)

```
for j ← 2 to length(A)
    do
        key ← A[j]
        ▶Insert A[j] into the sorted sequence A[1, ..., j − 1]}
        i ← j − 1
        while i > 0 and A[i] > key
            do
                A[i + 1] ← A[i]
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        A[i + 1] ← key
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Count Value

$\rightarrow \ c_1 N$

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$\rightarrow \quad c_4 \sum_{j=2}^{N} j$

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Count Value

$\rightarrow \; c_5 \sum_{j=2}^{N} (j-1)$

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Count Value

$\rightarrow \quad c_6 \sum_{j=2}^{N} (j-1)$

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Count Value

$\rightarrow \; c_7 \, (N-1)$

Outline

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The $T(N)$ function

The total number of operations

It is know as a function $T(N)$

$$T : \mathbb{N} \longmapsto \mathbb{N}$$

Something Notable

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Building a function for counting

Counting Equation

$$T(N) = c_1 N + c_2(N-1) + C_3(N-1) + c_4\left(\frac{N(N+1)}{2} - 1\right) + \ldots$$
$$c_5\left(\frac{N(N-1)}{2} - 1\right) + c_6\left(\frac{N(N-1)}{2} - 1\right) + c_7\left(N-1\right)$$

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Example of Complexities

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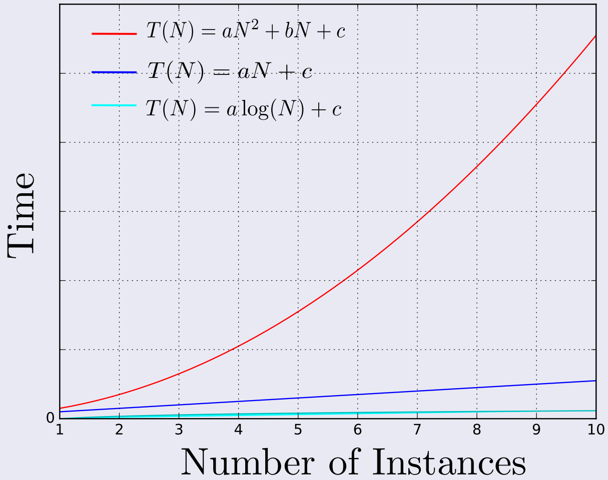

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The Worst and The Average Case Inputs

The Worst Case Input

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- In case of **insertion sort**, it will be the permutation:

$$N, N-1, N-2, ..., 3, 2, 1 \tag{1}$$

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