Introduction to Machine Learning

Universal Approximation Theorem of the Multilayer Perceptron

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Outline

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Introduction

Representation of functions

The main result in multi-layer perceptron is its power of representation.

Furthermore

After all, it is quite striking if we can represent continuous functions of the form $f : \mathbb{R}^n \longmapsto \mathbb{R}$ as a finite sum of simple functions.

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Therefore

Our main goal

We want to know under which conditions the sum of the form:

$$G(\boldsymbol{x}) = \sum_{j=1}^{N} \alpha_j f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta_j\right) \quad (1)$$

can represent continuous functions in a specific domain.

Setup of the problem

Definition of I_n

It is an n-dimensional unit cube $[0,1]^n$

In addition, we have the following set of functions

$$C(I_n) = \{f : I_n \to \mathbb{R} | f \text{ is a continous function}\} \quad (2)$$

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Topology

Definition (Topological Space)

A topological space is then a set X together with a collection of subsets of X, called **open sets** and satisfying the following axioms:

1. The empty set and X itself are open.
2. Any union of open sets is open.
3. The intersection of any finite number of open sets is open.

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Example

We have

- Given any set X, one can define a topology on X where every subset of X is an open set.

Also

- Let (X, d) be a metric space. The sets (called open Balls) are a Topology

$$S(x_0, r) = \{x \in X | d(x_0, x) < r\} \text{ where } r > 0 \text{ and } x_0 \in X$$

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Definition

Open Cover Definition

- A topological space X is called compact if each of its open covers has a finite subcover.

In Our Case

- I_n is compact

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Compactness

Theorem

A compact set is closed and bounded.

Thus

I_n is a compact set in $\mathbb{R}^n$.

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Why Compactness?

Basically

- Given that in Topology we care in how something behaves in open sets!!!

Compactness

- Establish some sort of "fitness" in a Topological sense

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- There are only finitely many possible behaviors.

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Continuous Functions

Theorem

- A function f from a topological space X into another topological space Y is continuous if and only if every open set V in Y,

$$f^{-1}(V) = \{x | f(x) \in V\}$$

Example. Is $f : [0,1] \longrightarrow \mathbb{R}$, $f(x) = 1/x$ a continuous function in $[0,1]$?

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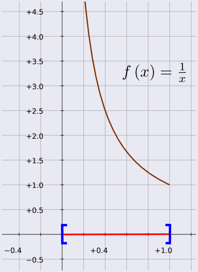

It is not!!

Define with g a continuous function

$$B_{\mathbb{R}}\left(\epsilon, g\left(x_0\right)\right) = \left\{y \in \mathbb{R} \mid \left\|y - f\left(x_0\right)\right\| < \epsilon\right\}$$

Therefore, its pre-image is open

$$f^{-1}\left(B_{\mathbb{R}}\left(\epsilon, g\left(x_0\right)\right)\right)$$

Therefore exist a ball around $B_{[0,1]}\left(\delta, x_0\right)$

$$B_{[0,1]}\left(\delta, x_0\right) \subseteq f^{-1}\left(B_{\mathbb{R}}\left(\epsilon, g\left(x_0\right)\right)\right)$$

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The well know $\epsilon - \delta$ definition

What about the $+\infty$ and our original f

- You need to use a sequence $\{x_n\}$ such that $x_n \to +\infty$ when $n \to \infty$

Therefore, we have for some $\epsilon >$

- We have that $\lim_{n\to\infty} B\left(\epsilon, f\left(x_n\right)\right) = \lim_{n\to\infty} x_n$

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$$\lim_{n\to\infty} f^{-1}\left(B_{\mathbb{R}}\left(\epsilon, g\left(x_n\right)\right)\right) = \{0\}$$

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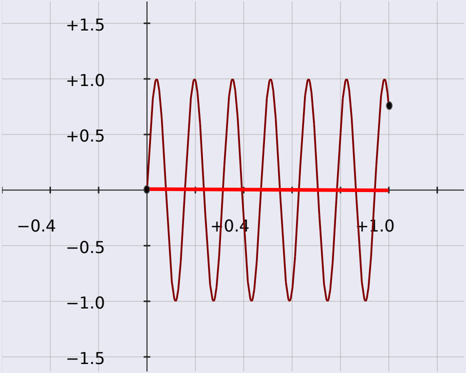
Therefore

The pre-image is closed

- The function f is not continuous!!!

All the continuous functions are bounded

For Example



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Theorem

- Let K be a nonempty subset of $\mathbb{R}^n$, where $n > 1$. If K is compact, then every continuous real-valued function defined on K is **bounded.**

Definition (Supremum Norm)

- Let X be a topological space and let F be the space of all bounded complex-valued continuous functions defined on K.
  - The supremum norm is the norm defined on F by

$$\|f\| = \sup_{x \in X} |f(x)| \tag{3}$$

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I give you an idea

We would like, given $C\left(I_n\right)$

- To prove that there is a function

$$\sum_{j=1}^{N} \alpha_j f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta_j\right)$$

Nearby any $f\left(\boldsymbol{x}\right) \in C\left[I_n\right]$

- Basically, we want a set

$$R = \left\{ G\left(\boldsymbol{x}\right) \middle| G\left(\boldsymbol{x}\right) = \sum_{j=1}^{N} \alpha_j f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta_j\right) \right\}$$

such that $R \subseteq C\left[I_n\right]$ and given $G \in R$, for all $\epsilon > 0$, $\sup_{\boldsymbol{x} \in I_n} \left|G\left(\boldsymbol{x}\right) - f\left(\boldsymbol{x}\right)\right| < \epsilon$.

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Limit Points

Definition

If X is a topological space and p is a point in X, a neighborhood of p is a subset V of X that includes an open set U containing p, $p \in U \subseteq V$.

- This is also equivalent to $p \in X$ being in the interior of V.

Example in a metric space

In a metric space (X, d), a set V is a neighborhood of a point p if there exists an open ball with center at p and radius $r > 0$, such that

$$B_r(p) = B(p; r) = \{x \in X | d(x, p) < r\} \tag{4}$$

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Limit Points

Definition of a Limit Point

Let S be a subset of a topological space X. A point $x \in X$ is a limit point of S if every neighborhood of x contains at least one point of S different from x itself.

Example in $\mathbb{R}$

Which are the limit points of the set $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$?

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This allows to define the idea of density

Something Notable

A subset A of a topological space X is dense in X, if for any point $x \in X$, any neighborhood of x contains at least one point from A.

Classic Example

The real numbers with the usual topology have the rational numbers as a countable dense subset.

- Why do you believe the floating-point numbers are rational?

In addition

Also the irrational numbers.

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From this, you have the idea of closure

Definition

The closure of a set S is the set of all points of closure of S, that is, the set S together with all of its limit points.

Example

The closure of the following set $(0,1) \cup \{2\}$

Meaning

Not all points in the closure are limit points.

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- We would love to be able to say that separation exist!!!
- Given two functions, we can say they are different if their mappings are different!!!

- So we can integrate them!!!

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Hausdorff Space

Definition of Separation

Points x and y in a topological space X can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint.

Definition

X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods.

This solve the first issue!!!

- We can identify different functions... by open sets

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Look at what we have

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We can integrate $f \in C(I_n)$

- However, we need to have a measure μ to integrate such functions

Why?

- We want to construct an existence theorem by contradiction and integration is necessary

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Now, the Measure Concept

Definition of $\sigma-$algebra

Let $\mathcal{A} \subset \mathcal{P}(X)$, we say that $\mathcal{A}$ to be an algebra if

1. $\emptyset, X \in \mathcal{A}$.
2. $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.
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Example

In $X = [0, 1)$, the class $\mathcal{A}_0$ consisting of $\emptyset$, and all finite unions $A = \cup_{i=1}^{n} [a_i, b_i)$ with $0 \leq a_i < b_i \leq a_{i+1} \leq 1$ is an algebra.

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Now, the Measure Concept

Definition of additivity

Let $\mu : \mathcal{A} \to [0, +\infty]$ be such that $\mu(\emptyset) = 0$, we say that μ is $\sigma-$additive if for any $\{A_i\}_{i \in I} \subset \mathcal{A}$ (Where I can be finite of infinite countable) of mutually disjoint sets such that $\cup_{i \in I} A_i \in \mathcal{A}$, we have that

$$\mu\left(\cup_{i \in I} A_i\right) = \sum_{i \in I} \mu\left(A_i\right) \quad (5)$$

Definition of Measurability

Let $\mathcal{A}$ be a $\sigma-$algebra of subsets of X, we say that the [air $(X, \mathcal{A})$ is a measurable space where a $\sigma-$additive function $\mu : \mathcal{A} \to [0, +\infty]$ is called a measure on $(X, \mathcal{A})$.

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Definition of a Borel Measure

If $\mathcal{F}$ is the Borel σ-algebra on some topological space, then a measure $\mu : \mathcal{F} \to \mathbb{R}$ is said to be a Borel measure (or Borel probability measure). For a Borel measure, all continuous functions are measurable.

1. $\mu(\emptyset) = 0$.
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3. $\sup_{A \in \mathcal{B}(X)} |\mu(A)| < \infty$

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A Borel Measure

Definition

The Borel σ-algebra is defined to be the σ-algebra generated by the open sets (or equivalently, by the closed sets).

Definition of a Borel Measure

If $\mathcal{F}$ is the Borel σ-algebra on some topological space, then a measure $\mu : \mathcal{F} \to \mathbb{R}$ is said to be a Borel measure (or Borel probability measure). For a Borel measure, all continuous functions are measurable.

Definition of a signed Borel Measure

A signed Borel measure $\mu : \mathcal{B}(X) \to$ is a measure such that

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Regularity

A measure μ is Borel regular measure:

1. For every Borel set $B \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$, $\mu(A) = \mu(A \cap B) + \mu(A - B)$.
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Outline

Discriminatory Functions

Definition

Given the set $M\left(I_n\right)$ of signed regular Borel measures, a function f is discriminatory if for a measure $\mu \in M\left(I_n\right)$

$$\int_{I_n} f\left(\boldsymbol{w}^T \boldsymbol{x} + \theta\right) d\mu = 0 \tag{6}$$

for all $\boldsymbol{w} \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$ **implies that** $\mu = 0$

Definition

We say that f is sigmoidal if

$$f(t) \to \begin{cases} 1 & \text{as } t \to +\infty \\ 0 & \text{as } t \to -\infty \end{cases}$$

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Outline

The Important Theorem

Theorem 1

Let f a be any continuous discriminatory function. Then finite sums of the form

$$G\left(\boldsymbol{x}\right)=\sum_{j=1}^{N}\alpha_{j}f\left(\boldsymbol{w}_{j}^{T}\boldsymbol{x}+\theta_{j}\right), \tag{7}$$

where $\boldsymbol{w}_j \in \mathbb{R}^n$ and α_j, $\theta_j \in \mathbb{R}$ are fixed, are dense in $C\left(I_n\right)$

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Meaning

In other words

Given any $g \in C(I_n)$ and $\epsilon > 0$, there is a sum, $G(\boldsymbol{x})$, of the above form, for which

$$|G(\boldsymbol{x}) - g(\boldsymbol{x})| < \epsilon \; \forall \boldsymbol{x} \in I_n \tag{8}$$

Proof

Let $S \subset C(I_n)$ be the set of functions of the form $G(\boldsymbol{x})$

First, S is a linear subspace of $C(I_n)$

Definition

A subset V of $\mathbb{R}^n$ is called a linear subspace of $\mathbb{R}^n$ if V contains the zero vector, and is closed under vector addition and scaling. That is, for $X, Y \in V$ and $c \in \mathbb{R}$, we have $X + Y \in V$ and $cX \in V$.

We claim that the closure of S is all of $C(I_n)$

Assume that the closure of S is not all of $C(I_n)$

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The closure of S, say R, is a **closed proper subspace** of $C(I_n)$

We use the Hahn-Banach Theorem

If $p : V \to \mathbb{R}$ is a sub-linear function (i.e. you have $p(x+y) \leq p(x) + p(y)$ and the product against scalar is the same), and $\varphi : U \to \mathbb{R}$ is a linear functional on a linear subspace $U \subseteq V$ which is dominated by p on U, i.e. $\varphi(x) \leq p(x)\ \forall x \in U$.

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Then

There exists a linear extension $\psi : V \to \mathbb{R}$ of φ to the whole space V, i.e., there exists a linear functional ψ such that

1. $\psi(x) = \varphi(x)\ \forall x \in U$.
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It is possible to construct sub-linear function defined as follow

We define the following linear functional

$$T(f) = \begin{cases} f & \text{if } f \in C(I_n) - R \\ 0 & \text{if } f \in R \end{cases} \quad (9)$$

Then

- Using T as p and φ
- $V = C(I_n)$
- $U = R$

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Therefore

We have

There is a bounded linear functional called $L \neq 0$

- The ψ in the Hahn-Banach Theorem
- With $L(R) = L(S) = 0$, but $L(C(I_n) - R) \neq 0$

Proof

Now, we use the Riesz Representation Theorem

Let X be a locally compact Hausdorff space. For any positive linear functional ψ on $C(X)$, there is a unique regular Borel measure μ on X such that

$$\psi = \int_X f(x)\, d\mu(x) \tag{10}$$

for all f in $C(X)$

We can then do the following

$$L(h) = \int_{I_n} h(x)\, d\mu(x) \tag{11}$$

Where?

For some $\mu \in M(I_n)$, for all $h \in C(I_n)$

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In particular

Given that $f\left(\boldsymbol{w}^T\boldsymbol{x}+\theta\right)$ is in R for all $\boldsymbol{w}$ and θ

We must have that

$$\int_{I_n} f\left(\boldsymbol{w}^T\boldsymbol{x}+\theta\right) d\mu\left(\boldsymbol{x}\right) = 0 \tag{12}$$

for all w and θ

But we assumed that f is discriminatory!!!

- Then... $\mu = 0$ contradicting the fact that $L \neq 0$!!! In $f \in C\left(I_n\right) - R$
- We have a contradiction!!!

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Proof

Finally

The subspace S of sums of the form G is dense!!!

Now, we deal with the sigmoidal function

Lemma 1

Any bounded, measurable sigmoidal function, f, is discriminatory. In particular, any continuous sigmoidal function is discriminatory.

Proof

I will leave this to you... it is possible I will get a question from this proof for the firs midterm.

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We have the theorem finally!!!

Universal Representation Theorem for the multi-layer perceptron

Let f be any continuous sigmoidal function. Then finite sums of the form

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In other words

Given any $g \in C\left(I_{n}\right)$ and $\epsilon > 0$, there is a sum $G\left(x\right)$ of the above form, for which

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Proof

Simple

Combine the theorem and lemma 1... and because the continuous sigmoidals satisfy the conditions of the lemma

- We have our representation!!!