Introduction to Artificial Intelligence

Introduction Single-Layer Perceptron

Andres Mendez-Vazquez

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Outline

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History

At the beginning of Neural Networks (1943 - 1958)

- McCulloch and Pitts (1943) for introducing the idea of neural networks as computing machines.
- Hebb (1949) for postulating the first rule for self-organized learning.
- Rosenblatt (1958) for proposing the perceptron as the first model for learning with a teacher (i.e., supervised learning).

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Something Notable

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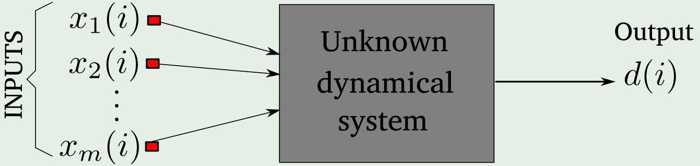
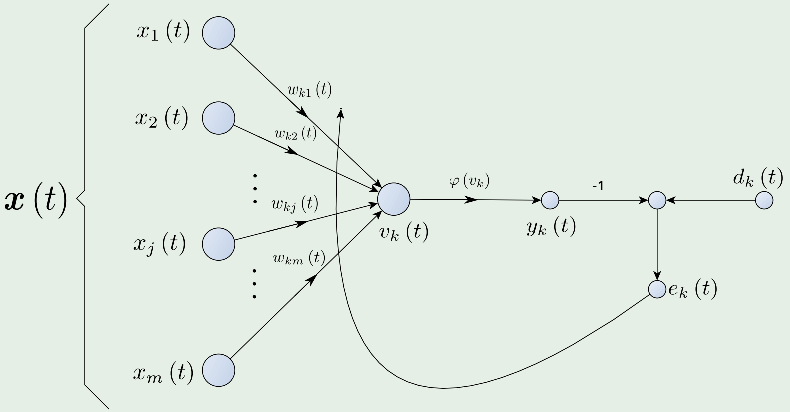
Adapting Filtering Problem

Consider a dynamical system



Signal-Flow Graph of Adaptive Model

We have the following equivalence



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Description of the Behavior of the System

We have the data set

$$\mathcal{T} = \left\{ \left(\boldsymbol{x}\left(i\right), d\left(i\right) \right) | i = 1, 2, ..., n, ... \right\} \tag{1}$$

Where

$$\boldsymbol{x}\left(i\right) = \left(x_1\left(i\right), x_2\left(i\right) ..., x_m\left(i\right)\right)^T \tag{2}$$

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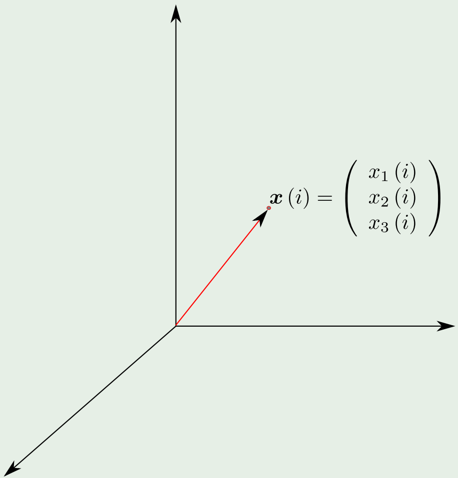
The Stimulus $\boldsymbol{x}(i)$

The stimulus $\boldsymbol{x}(i)$ can arise from

The m elements of $\boldsymbol{x}(i)$ originate at different points in space (spatial)



The Stimulus $\boldsymbol{x}(i)$

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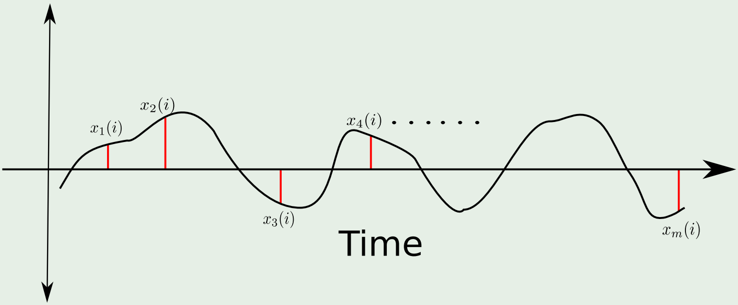
The m elements of $\boldsymbol{x}(i)$ represent the set of present and $(m-1)$ past values of some excitation that are uniformly spaced in time (temporal).



Problem

Quite important

How do we design a multiple input-single output model of the unknown dynamical system?

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We want to build this around a single neuron!!!

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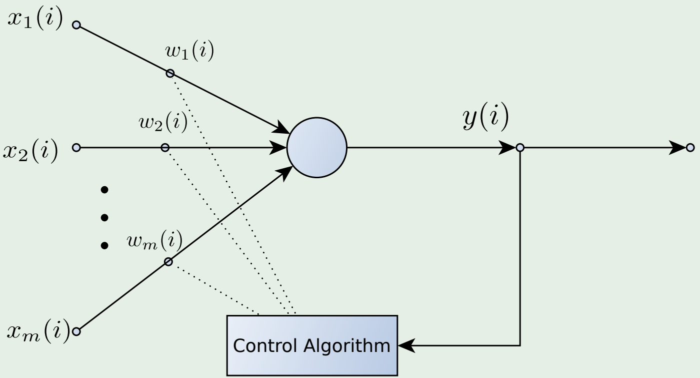
How do we design a multiple input-single output model of the unknown dynamical system?

It is more

We want to build this around a single neuron!!!

Thus, we have the following...

We need an algorithm to control the weight adjustment of the neuron



Which steps do you need for the algorithm?

First

The algorithms starts from an arbitrary setting of the neuron's synaptic weight.

Second

Adjustments, with respect to changes on the environment, are made on a continuous basis.

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First

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Adjustments, with respect to changes on the environment, are made on a continuous basis.

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Third

Computation of adjustments to synaptic weights are completed inside a time interval that is one sampling period long.

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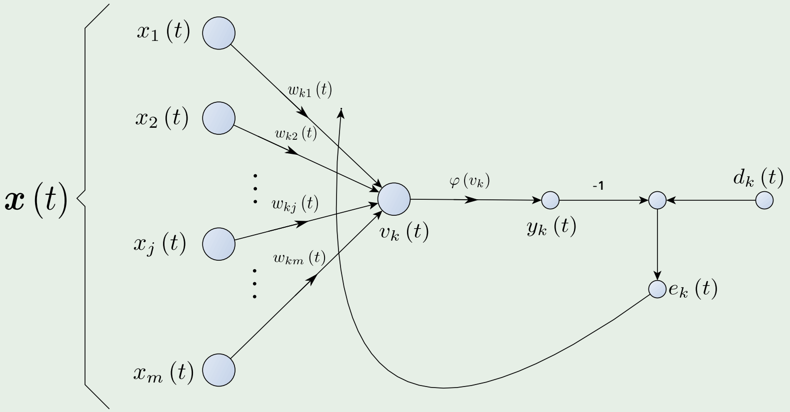

Thus, This Neural Model $\approx$ Adaptive Filter with two continous processes

Filtering processes

1. An output, denoted by $y(i)$, that is produced in response to the m elements of the stimulus vector $\boldsymbol{x}(i)$.
2. An error signal, $e(i)$, that is obtained by comparing the output $y(i)$ to the corresponding desired output $d(i)$ produced by the unknown system.

Adaptive Process

It involves the automatic adjustment of the synaptic weights of the neuron in accordance with the error signal $e(i)$

Remark

The combination of these two processes working together constitutes a **feedback loop** acting around the neuron.

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Thus

The output $y(i)$ is exactly the same as the induced local field $v(i)$

$$y(i) = v(i) = \sum_{i=1}^{m} w_k(i)\, x_k(i) \tag{3}$$

In matrix form, we have - remember we only have a neuron, so we do not have neuron k

$$y(i) = \boldsymbol{x}^T(i)\, \boldsymbol{w}(i) \tag{4}$$

Error

$$e(i) = d(i) - y(i) \tag{5}$$

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Consider

A continous differentiable function $J(\boldsymbol{w})$

We want to find an optimal solution $\boldsymbol{w}^*$ such that

$$J(\boldsymbol{w}^*) \leq J(\boldsymbol{w}), \ \forall \boldsymbol{w} \tag{6}$$

We want to

Minimize the cost function $J(w)$ with respect to the weight vector w.

For this

$$\nabla J(w) = 0 \tag{7}$$

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∇ is the gradient operator

$$\nabla = \left[\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, ..., \frac{\partial}{\partial w_m} \right]^T \tag{8}$$

Thus

$$\nabla J(w) = \left[\frac{\partial J(w)}{\partial w_1}, \frac{\partial J(w)}{\partial w_2}, ..., \frac{\partial J(w)}{\partial w_m} \right]^T \tag{9}$$

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Thus

Starting with an initial guess denoted by $\boldsymbol{w}(0)$,

Then, generate a sequence of weight vectors $\boldsymbol{w}(1), \boldsymbol{w}(2), ...$

Such that you can reduce $J(\boldsymbol{w})$ at each iteration

$$J(\boldsymbol{w}(n+1)) < J(\boldsymbol{w}(n)) \tag{10}$$

Where: $\boldsymbol{w}(n)$ is the old value and $\boldsymbol{w}(n+1)$ is the new value.

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The Three Main Methods for Unconstrained Optimization

We will look at

1. Steepest Descent.
2. Newton's Method
3. Gauss-Newton Method

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Steepest Descent

In the method of steepest descent, we have a cost function $J(\boldsymbol{w})$ where

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) - \eta \nabla J(\boldsymbol{w}(\boldsymbol{n}))$$

How, we prove that $J(\boldsymbol{w}(n+1)) < J(\boldsymbol{w}(n))$?

We use the first-order Taylor series expansion around $\boldsymbol{w}(n)$

$$J(\boldsymbol{w}(n+1)) \approx J(\boldsymbol{w}(n)) + \nabla J^T(\boldsymbol{w}(\boldsymbol{n}))\Delta \boldsymbol{w}(n) \quad (11)$$

Remark: This is quite true when the step size is quite small!!! In addition, $\Delta \boldsymbol{w}(n) = \boldsymbol{w}(n+1) - \boldsymbol{w}(n)$

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Why? Look at the case in $\mathbb{R}$

The equation of the tangent line to the curve $y = J(w(n))$

$$L(w(n)) = J'(w(n))\left[w(n+1) - w(n)\right] + J(w(n)) \tag{12}$$

Example

Why? Look at the case in $\mathbb{R}$

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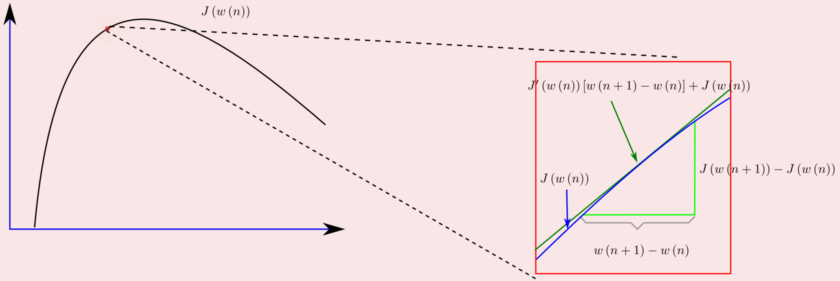
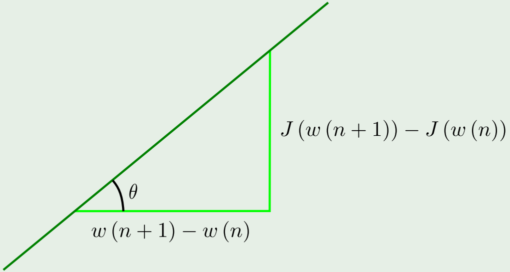


Thus, we have that in $\mathbb{R}$

Remember Something quite Classic



$$\tan\theta = \frac{J(w(n+1)) - J(w(n))}{w(n+1) - w(n)}$$

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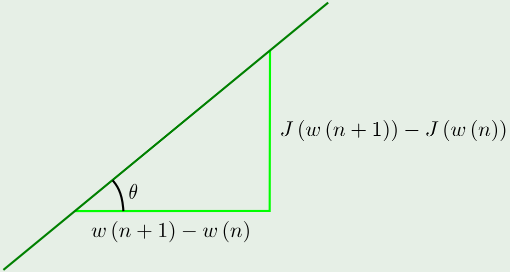

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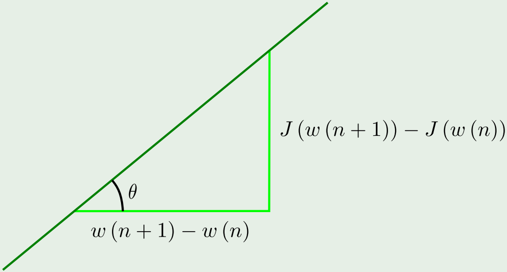

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Thus, we have that

Using the First Taylor expansion

$$J\left(w\left(n\right)\right) \approx J\left(w\left(n\right)\right) + J'\left(w\left(n\right)\right)\left[w\left(n+1\right) - w\left(n\right)\right] \quad (13)$$

Now, for Many Variables

An hyperplane in $\mathbb{R}^n$ is a set of the form

$$H = \left\{ \boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} = b \right\} \tag{14}$$

Given $\boldsymbol{x} \in H$ and $\boldsymbol{x}_0 \in H$

$$b = \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}^T \boldsymbol{x}_0$$

Thus, we have that

$$H = \left\{ \boldsymbol{x} | \boldsymbol{a}^T (\boldsymbol{x} - \boldsymbol{x}_0) = 0 \right\}$$

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Thus, we have the following definition

Definition (Differentiability)

Assume that J is defined in a disk D containing $\boldsymbol{w}(n)$. We say that J is differentiable at $\boldsymbol{w}(n)$ if:

1. $\frac{\partial J(\boldsymbol{w}(n))}{\partial w_i}$ exist for all $i = 1, ..., n$.
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Thus, given $J(\boldsymbol{w}(n))$

We know that we have the following operator

$$\nabla = \left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, ..., \frac{\partial}{\partial w_m}\right) \quad (15)$$

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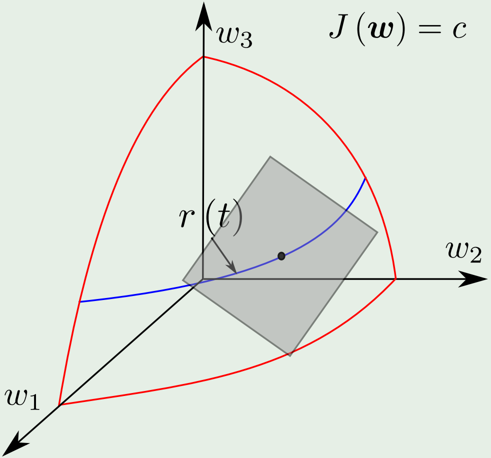
Thus, we have

$$\begin{aligned}\nabla J\left(\boldsymbol{w}\left(n\right)\right) &= \left(\frac{\partial J\left(\boldsymbol{w}\left(n\right)\right)}{\partial w_1}, \frac{\partial J\left(\boldsymbol{w}\left(n\right)\right)}{\partial w_2}, ..., \frac{\partial J\left(\boldsymbol{w}\left(n\right)\right)}{\partial w_m}\right)\\ &= \sum_{i=1}^{m} \hat{w}_i \frac{\partial J\left(\boldsymbol{w}\left(n\right)\right)}{\partial w_i}\end{aligned}$$

Where: $\hat{w}_i^T = (1, 0, ..., 0) \in \mathbb{R}$

Now

Given a curve function $r(t)$ that lies on the level set $J(\boldsymbol{w}(n)) = c$ (When is in $\mathbb{R}^3$)



Level Set

Definition

$$\{(w_1, w_2, ..., w_m) \in \mathbb{R}^m | J(w_1, w_2, ..., w_m) = c\} \quad (16)$$

Remark: In a normal Calculus course we will use x and f instead of $\boldsymbol{w}$ and J.

Where

Any curve has the following parametrization

$$r:[a,b]\to\mathbb{R}^m$$
$$r(t)=\left(w_1\left(t\right),...,w_m\left(t\right)\right)$$

With $r(n+1)=\left(w_1\left(n+1\right),...,w_m\left(n+1\right)\right)$

We can write the parametrized version of it

$$z(t)=J\left(w_1\left(t\right),w_2\left(t\right),...,w_m\left(t\right)\right)=c \quad (17)$$

Differentiating with respect to t and using the chain rule for multiple variables

$$\frac{dz(t)}{dt}=\sum_{i=1}^{m}\frac{\partial J\left(\boldsymbol{w}\left(t\right)\right)}{\partial w_i}\cdot\frac{dw_i(t)}{dt}=0 \quad (18)$$

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Note

First

Given $y = f(\boldsymbol{u}) = (f_1(\boldsymbol{u}), ..., f_l(\boldsymbol{u}))$ and $\boldsymbol{u} = g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), ..., g_m(\boldsymbol{x}))$.

We have then that

$$\frac{\partial(f_1, f_2, ..., f_l)}{\partial(x_1, x_2, ..., x_k)} = \frac{\partial(f_1, f_2, ..., f_l)}{\partial(g_1, g_2, ..., g_m)} \cdot \frac{\partial(g_1, g_2, ..., g_m)}{\partial(x_1, x_2, ..., x_k)} \quad (19)$$

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First

Given $y = f(\boldsymbol{u}) = (f_1(\boldsymbol{u}), ..., f_l(\boldsymbol{u}))$ and $\boldsymbol{u} = g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), ..., g_m(\boldsymbol{x}))$.

We have then that

$$\frac{\partial(f_1, f_2, ..., f_l)}{\partial(x_1, x_2, ..., x_k)} = \frac{\partial(f_1, f_2, ..., f_l)}{\partial(g_1, g_2, ..., g_m)} \cdot \frac{\partial(g_1, g_2, ..., g_m)}{\partial(x_1, x_2, ..., x_k)} \tag{19}$$

Thus

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Thus

Evaluating at $t = n$

$$\sum_{i=1}^{m} \frac{\partial J\left(\boldsymbol{w}\left(n\right)\right)}{\partial w_i} \cdot \frac{dw_i(n)}{dt} = 0$$

We have that

$$\nabla J\left(\boldsymbol{w}\left(n\right)\right) \cdot \boldsymbol{r}'\left(n\right) = 0 \tag{20}$$

This proves that for every level set the gradient is perpendicular to the tangent to any curve that lies on the level set

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Now the tangent plane to the surface can be described generally

Thus

$$L(\boldsymbol{w}(n+1)) = J(\boldsymbol{w}(n)) + \nabla J^T(\boldsymbol{w}(\boldsymbol{n}))[\boldsymbol{w}(n+1) - \boldsymbol{w}(n)] \qquad (21)$$

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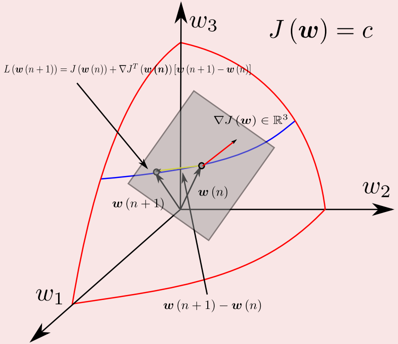

Proving the fact about the Steepest Descent

We want the following

$$J\left(\boldsymbol{w}\left(n+1\right)\right) < J\left(\boldsymbol{w}\left(n\right)\right)$$

Using the first-order Taylor approximation

$$J\left(\boldsymbol{w}\left(n+1\right)\right) - J\left(\boldsymbol{w}\left(n\right)\right) \approx \nabla J^{T}\left(\boldsymbol{w}\left(n\right)\right)\Delta\boldsymbol{w}\left(n\right)$$

So, we ask the following

$$\Delta\boldsymbol{w}\left(n\right) \approx -\eta\nabla J\left(\boldsymbol{w}\left(n\right)\right) \text{ with } \eta > 0$$

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Thus

$$J\left(\boldsymbol{w}\left(n+1\right)\right)-J\left(\boldsymbol{w}\left(n\right)\right)<0$$

Or

$$J\left(\boldsymbol{w}\left(n+1\right)\right)<J\left(\boldsymbol{w}\left(n\right)\right)$$

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Outline

Newton's Method

Here

The basic idea of Newton's method is to minimize the quadratic approximation of the cost function $J(\boldsymbol{w})$ around the current point $\boldsymbol{w}(n)$.

Using a second-order Taylor series expansion of the cost function around the point $\boldsymbol{w}(n)$

$$\begin{aligned}\Delta J(\boldsymbol{w}(n)) &= J(\boldsymbol{w}(n+1)) - J(\boldsymbol{w}(n)) \\ &\approx \nabla J^T(\boldsymbol{w}(\boldsymbol{n}))\Delta\boldsymbol{w}(n) + \frac{1}{2}\Delta\boldsymbol{w}^T(n)H(n)\Delta\boldsymbol{w}(n)\end{aligned}$$

Where given that $\boldsymbol{w}(n)$ is a vector with dimension m

$$H = \nabla^2 J(\boldsymbol{w}) = \begin{pmatrix} \frac{\partial^2 J(\boldsymbol{w})}{\partial w_1^2} & \frac{\partial^2 J(\boldsymbol{w})}{\partial w_1 \partial w_2} & \cdots & \frac{\partial^2 J(\boldsymbol{w})}{\partial w_1 \partial w_m} \\ \frac{\partial^2 J(\boldsymbol{w})}{\partial w_2 \partial w_1} & \frac{\partial^2 J(\boldsymbol{w})}{\partial w_2^2} & \cdots & \frac{\partial^2 J(\boldsymbol{w})}{\partial w_2 \partial w_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 J(\boldsymbol{w})}{\partial w_m \partial w_1} & \frac{\partial^2 J(\boldsymbol{w})}{\partial w_m \partial w_2} & \cdots & \frac{\partial^2 J(\boldsymbol{w})}{\partial w_m^2} \end{pmatrix}$$

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Now, we want to minimize $J\left(\boldsymbol{w}\left(n+1\right)\right)$

Do you have any idea?

Look again

$$J\left(\boldsymbol{w}\left(n\right)\right)+\nabla J^{T}\left(\boldsymbol{w}\left(\boldsymbol{n}\right)\right)\Delta\boldsymbol{w}\left(n\right)+\frac{1}{2}\Delta\boldsymbol{w}^{T}\left(n\right)H\left(n\right)\Delta\boldsymbol{w}\left(n\right) \quad (22)$$

Derive with respect to $\Delta\boldsymbol{w}\left(n\right)$

$$\nabla J\left(\boldsymbol{w}\left(\boldsymbol{n}\right)\right)+H\left(n\right)\Delta\boldsymbol{w}\left(n\right)=0 \quad (23)$$

Thus

$$\Delta\boldsymbol{w}\left(n\right)=-H^{-1}\left(n\right)\nabla J\left(\boldsymbol{w}\left(\boldsymbol{n}\right)\right)$$

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The Final Method

Define the following

$$J\left(\boldsymbol{w}\left(n+1\right)\right)-J\left(\boldsymbol{w}\left(n\right)\right)=-H^{-1}\left(n\right)\nabla J\left(\boldsymbol{w}\left(\boldsymbol{n}\right)\right)$$

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Outline

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We have then an error

Something Notable

$$J(\boldsymbol{w}) = \frac{1}{2}\sum_{i=1}^{n} e^2(i)$$

Thus using the first order Taylor expansion, we linearize

$$e_l(i, \boldsymbol{w}) = e(i) + \left[\frac{\partial e(i)}{\partial \boldsymbol{w}}\right]^T [\boldsymbol{w} - \boldsymbol{w}(n)]$$

In matrix form

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Where

The error vector is equal to

$$\boldsymbol{e}(n) = [e(1), e(2), ..., e(n)]^T \tag{24}$$

Thus, we get the famous Jacobian once we derive $\frac{\partial e(i)}{\partial w}$

$$J(n) = \begin{pmatrix} \frac{\partial e(1)}{\partial w_1} & \frac{\partial e(1)}{\partial w_2} & \cdots & \frac{\partial e(1)}{\partial w_m} \\ \frac{\partial e(2)}{\partial w_1} & \frac{\partial e(2)}{\partial w_2} & \cdots & \frac{\partial e(2)}{\partial w_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial e(n)}{\partial w_1} & \frac{\partial e(n)}{\partial w_2} & \cdots & \frac{\partial e(n)}{\partial w_m} \end{pmatrix}$$

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What if we expand out the equation?

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Ideas

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Expanded Version

We get

$$\frac{1}{2}\left\|\boldsymbol{e}_l\left(n,\boldsymbol{w}\right)\right\|^2 = \frac{1}{2}\left\|\boldsymbol{e}\left(n\right)\right\|^2 + \boldsymbol{e}^T\left(n\right)\boldsymbol{J}\left(n\right)\left(\boldsymbol{w}-\boldsymbol{w}\left(n\right)\right) + \ldots$$
$$\frac{1}{2}\left(\boldsymbol{w}-\boldsymbol{w}\left(n\right)\right)^T\boldsymbol{J}^T\left(n\right)\boldsymbol{J}\left(n\right)\left(\boldsymbol{w}-\boldsymbol{w}\left(n\right)\right)$$

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Now,doing the Differential, we get

Differentiating the equation with respect to $\boldsymbol{w}$

$$\mathsf{J}^{T}(n)\,\boldsymbol{e}(n) + \mathsf{J}^{T}(n)\,\mathsf{J}(n)\left[\boldsymbol{w} - \boldsymbol{w}(n)\right] = 0$$

We get finally

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) - \left(\mathsf{J}^{T}(n)\,\mathsf{J}(n)\right)^{-1}\mathsf{J}^{T}(n)\,\boldsymbol{e}(n) \qquad (25)$$

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Remarks

We have that

- The Newton's method that requires knowledge of the Hessian matrix of the cost function.
- The Gauss-Newton method only requires the Jacobian matrix of the error vector $\boldsymbol{e}(n)$.

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The Gauss-Newton iteration to be computable, the matrix product $\mathbf{J}^T(n)\,\mathbf{J}(n)$ must be nonsingular!!!

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Outline

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Introduction

A linear least-squares filter has two distinctive characteristics

- First, the single neuron around which it is built is linear.
- The cost function $J(\boldsymbol{w})$ used to design the filter consists of the sum of error squares.

Thus, expressing the error

$$e(n) = d(n) - (\boldsymbol{x}(1), \ldots, \boldsymbol{x}(n))^T \boldsymbol{w}(n)$$

Short Version - error is linear in the weight vector $\boldsymbol{w}(n)$

$$e(n) = d(n) - \boldsymbol{X}(n)\boldsymbol{w}(n)$$

- Where $d(n)$ is a $n \times 1$ desired response vector.
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Now, differentiate $e(n)$ with respect to $\boldsymbol{w}(n)$

Thus

$$\nabla e(n) = -\boldsymbol{X}^T(n)$$

Correspondingly, the Jacobian of $e(n)$ is

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Let us to use the Gaussian-Newton

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) - \left(\mathsf{J}^T(n)\,\mathsf{J}(n)\right)^{-1}\mathsf{J}^T(n)\,e(n)$$

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We have then

$$\boldsymbol{w}(n+1) = \boldsymbol{w}(n) + \left(\boldsymbol{X}^T(n)\boldsymbol{X}(n)\right)^{-1}\boldsymbol{X}^T(n)d(n) - \dots$$
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$$\boldsymbol{w}(n+1)=\boldsymbol{w}(n)-\left(-\boldsymbol{X}^{T}(n)\cdot-\boldsymbol{X}(n)\right)^{-1}\cdot-\boldsymbol{X}^{T}(n)\left[d(n)-\boldsymbol{X}(n)\boldsymbol{w}(n)\right]$$

We have then

$$\begin{aligned}\boldsymbol{w}(n+1)=&\boldsymbol{w}(n)+\left(\boldsymbol{X}^{T}(n)\boldsymbol{X}(n)\right)^{-1}\boldsymbol{X}^{T}(n)d(n)-...\\&\left(\boldsymbol{X}^{T}(n)\boldsymbol{X}(n)\right)^{-1}\boldsymbol{X}^{T}(n)\boldsymbol{X}(n)\boldsymbol{w}(n)\end{aligned}$$

Thus, we have

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Outline

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Again Our Error Cost function

We have

$$J(\boldsymbol{w}) = \frac{1}{2} e^2(n)$$

where $e(n)$ is the error signal measured at time n.

Again differentiating against the vector $\boldsymbol{w}$

$$\frac{\partial J(\boldsymbol{w})}{\partial \boldsymbol{w}} = e(n) \frac{\partial e(n)}{\partial \boldsymbol{w}}$$

LMS algorithm operates with a linear neuron so we may express the error signal as

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$$\frac{\partial e(n)}{\partial \boldsymbol{w}} = -\boldsymbol{x}(n)$$

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Remarks

The feedback loop around the weight vector low-pass filter

- It behaves like a low-pass filter.
- It passes the low frequency component of the error signal and attenuating its high frequency component.

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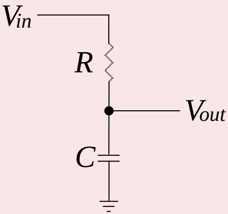

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Outline

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Convergence of the LMS

This convergence depends on the following points

- The statistical characteristics of the input vector $\boldsymbol{x}(n)$.
- The learning-rate parameter η.

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However instead using $E[\hat{\boldsymbol{w}}(n)]$ as $n \to \infty$, we use $E[e^2(n)] \to$constant as $n \to \infty$

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To make this analysis practical

We take the following assumptions

- The successive input vectors $\boldsymbol{x}(1), \boldsymbol{x}(2), ..$ are statistically independent of each other.
- At time step n, the input vector $\boldsymbol{x}(n)$ is statistically independent of all previous samples of the desired response, namely $d(1), d(2), ..., d(n-1)$.
- At time step n, the desired response $d(n)$ is dependent on $\boldsymbol{x}(n)$, but statistically independent of all previous values of the desired response.
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Because $\lambda_{\max}$ is the largest eigenvalue of the correlation sample $\boldsymbol{R}_x$

This can be difficult in reality.... then we use the trace instead

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Virtues and Limitations of the LMS Algorithm

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More of this in...

Simon Haykin

Simon Haykin - Adaptive Filter Theory (3rd Edition)

Exercises

We have from NN by Haykin

3.1, 3.2, 3.3, 3.4, 3.5, 3.7 and 3.8

Outline

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Objective

Goal

Correctly classify a series of samples (External applied stimuli) $x_1, x_2, x_3, ..., x_m$ into one of two classes, C_1 and C_2.

Output of each input

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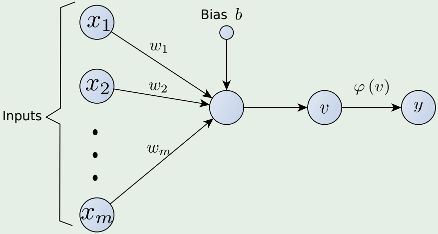
Perceptron: Local Field of a Neuron

Signal-Flow



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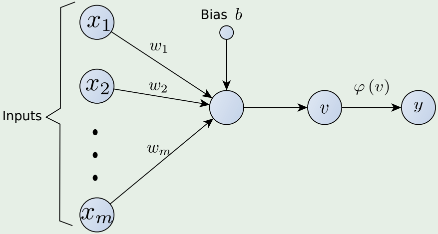

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Outline

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Based in the previous induced local field

In the simplest form of the perceptron there are two decision regions separated by an **hyperplane:**

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Example with two signals

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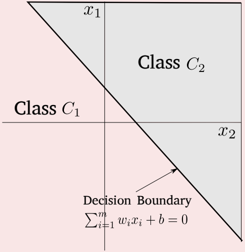

Outline

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Deriving the Algorithm

First, you put signals together

$$x(n) = [1, x_1(n), x_2(n), ..., x_m(n)]^T \tag{33}$$

Weights

$$v(n) = \sum_{i=0}^{m} w_i(n) x_i(n) = w^T(n) x(n) \tag{34}$$

Note IMPORTANT - Perceptron works only if C_1 and C_2 are linearly separable

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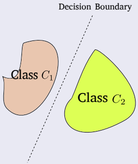

Rule for Linear Separable Classes

There must exist a vector $\boldsymbol{w}$

1. $\boldsymbol{w}^T\boldsymbol{x} > 0$ for every input vector $\boldsymbol{x}$ belonging to class C_1.
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A little bit on the Geometry

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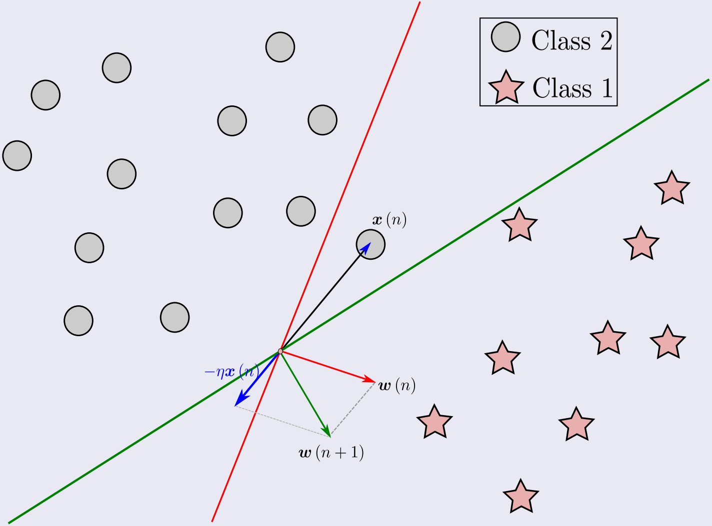

Outline

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Under Linear Separability - Convergence happens!!!

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Rosenblatt - 1962

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PERCEPTRON INCORRECTLY CLASSIFY THE VECTORS $\boldsymbol{x}\left(1\right),\boldsymbol{x}\left(2\right),...$

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Now assume for time $n = 1, 2, 3, \ldots$

$$\boldsymbol{w}^T(n)\,\boldsymbol{x}(n) < 0 \tag{38}$$

with $\boldsymbol{x}(n)$ belongs to class C_1.

PERCEPTRON INCORRECTLY CLASSIFY THE VECTORS $\boldsymbol{x}(1), \boldsymbol{x}(2), \ldots$

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Proof I - First a Lower Bound for $\|\boldsymbol{w}(n+1)\|^2$

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Apply the correction iteratively

$$\boldsymbol{w}(n+1) = \boldsymbol{x}(1) + \boldsymbol{x}(2) + \ldots + \boldsymbol{x}(n) \tag{40}$$

We know that there is a solution w_0(Linear Separability)

$$\alpha = \min_{\boldsymbol{x}(n) \in C_1} \boldsymbol{w}_0^T \boldsymbol{x}(n) \tag{41}$$

Then, we have

$$\boldsymbol{w}_0^T \boldsymbol{w}(n+1) = \boldsymbol{w}_0^T \boldsymbol{x}(1) + \boldsymbol{w}_0^T \boldsymbol{x}(2) + \ldots + \boldsymbol{w}_0^T \boldsymbol{x}(n) \tag{42}$$

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Proof III

Apply the correction iteratively

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Proof IV

Thus we use the α

$$\boldsymbol{w}_{\boldsymbol{0}}^{\boldsymbol{T}}\boldsymbol{w}\left(n+1\right)\geq n\alpha \quad (46)$$

Thus using the Cauchy-Schwartz Inequality

$$\left\|w_0^T\right\|^2\left\|w\left(n+1\right)\right\|^2\geq\left[w_0^T w\left(n+1\right)\right]^2 \quad (47)$$

$\|\cdot\|$ is the Euclidean distance.

Thus

$$\left\|w_0^T\right\|^2\left\|w\left(n+1\right)\right\|^2 \geq n^2\alpha^2$$

$$\left\|w\left(n+1\right)\right\|^2 \geq \frac{n^2\alpha^2}{\left\|w_0^T\right\|^2}$$

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Proof V - Now a Upper Bound for $\|\boldsymbol{w}(n+1)\|^2$

Now rewrite equation 39

$$\boldsymbol{w}(k+1) = \boldsymbol{w}(k) + \boldsymbol{x}(k) \tag{48}$$

for $k = 1, 2, ..., n$ and $\boldsymbol{x}(k) \in C_1$

Squaring the Euclidean norm of both sides

$$\|\boldsymbol{w}(k+1)\|^2 = \|\boldsymbol{w}(k)\|^2 + \|\boldsymbol{x}(k)\|^2 + 2\boldsymbol{w}^T(k)\boldsymbol{x}(k) \tag{49}$$

Now taking that $\boldsymbol{w}^T(k)\boldsymbol{x}(k) < 0$

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Proof VI

Use the telescopic sum

$$\sum_{k=0}^{n}\left[\|\boldsymbol{w}(k+1)\|^2-\|\boldsymbol{w}(k)\|^2\right]\leq\sum_{k=0}^{n}\|\boldsymbol{x}(k)\|^2 \tag{50}$$

Assume

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$$\|\boldsymbol{w}(n+1)\|^2 \leq \sum_{k=1}^{n}\|\boldsymbol{x}(k)\|^2$$

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Proof VII

Then, we can define a positive number

$$\beta = \max_{\boldsymbol{x}(k) \in C_1} \|\boldsymbol{x}(k)\|^2 \tag{51}$$

Thus

$$\|w(k+1)\|^2 \leq \sum_{k=1}^{n} \|x(k)\|^2 \leq n\beta$$

Thus, we satisfies the equations only when exists a n_{max} (Using Our Sandwich)

$$\frac{n_{max}^2 \alpha^2}{\|w_0\|^2} = n_{max}\beta \tag{52}$$

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Proof VIII

Solving

$$n_{max} = \frac{\beta \left\| \boldsymbol{w}_0 \right\|^2}{\alpha^2} \quad (53)$$

Thus

For $\eta(n) = 1$ for all n, $w(0) = 0$ and a solution vector w_0:

- The rule for adaptying the synaptic weights of the perceptron must terminate after at most n_{max} steps.

In addition

Because w_0 the solution is not unique.

Proof VIII

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Outline

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Algorithm Using Error-Correcting

Now, if we use the $\frac{1}{2}e_k(n)^2$

We can actually simplify the rules and the final algorithm!!!

Thus, we have the following Delta Value

$$\Delta w(n) = \eta((d_j - y_j(n)))x(n) \tag{54}$$

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Outline

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We could use the previous Rule

In order to generate an algorithm

However, you need classes that are linearly separable!!!

Therefore, we can use a more generals Gradient Descent Rule

To obtain an algorithm to the best separation hyperplane!!!

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Gradient Descent Algorithm

Algorithm - Off-line/Batch Learning

1. Set $n = 0$.
2. Set $d_j = \begin{cases} +1 & \text{if } x_j(n) \in \text{Class 1} \\ -1 & \text{if } x_j(n) \in \text{Class 2} \end{cases}$ for all $j = 1, 2, ..., m$.
3. Initialize the weights, $w^T = (w_1(n), w_2(n), ..., w_n(n))$.
   - Weights may be initialized to 0 or to a small random value.
4. Initialize Dummy outputs so you can enter loop $y^t = \langle y_1(n), y_2(n), ..., y_m(n)\rangle$
5. Initialize Stopping error $\epsilon > 0$.
6. Initialize learning error η.
7. While $\frac{1}{m}\sum_{j=1}^{m} \|d_j - y_j(n)\| > \epsilon$
   - For each sample (x_j, d_j) for $j = 1, ..., m$:
     - Calculate output $y_j = \varphi\left(\boldsymbol{w}^T(n) \cdot \boldsymbol{x}_j\right)$
     - Update weights $w_i(n+1) = w_i(n) + \eta(d_j - y_j(n))\, x_{ij}$.
   - $n = n + 1$

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We have the following problem

$$\epsilon > 0 \quad (55)$$

Thus...

Convergence to the best linear separation is a tweaking business!!!

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Outline

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However, if we limit our features!!!

The Winnow Algorithm!!!

It converges even with no-linear separability.

Feature Vector

A Boolean-valued features $X = \{0, 1\}^d$

Weight Vector w

1. $w^t = (w_1, w_2, \ldots, w_p)$ for all $w_i \in \mathbb{R}$
2. For all i, $w_i \geq 0$.

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Classification Scheme

We use a specific θ

1. $\boldsymbol{w}^T\boldsymbol{x} \geq \theta \Rightarrow$ positive classification Class 1 if $\boldsymbol{x}$ $\in$Class 1
2. $\boldsymbol{w}^T\boldsymbol{x} < \theta \Rightarrow$ negative classification Class 2 if $\boldsymbol{x}$ $\in$Class 2

Rule

We use two possible Rules for training!!! With a learning rate of $\alpha > 1$.

Rule 1

- When misclassifying a positive training example $\boldsymbol{x}$ $\in$Class 1 i.e. $\boldsymbol{w}^T\boldsymbol{x} < \theta$

$$\forall x_i = 1 : w_i \leftarrow \alpha w_i \quad (56)$$

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Classification Scheme

Rule 2

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$$\forall x_i = 1 : w_i \leftarrow \frac{w_i}{\alpha} \tag{57}$$

Rule 3

- If samples are correctly classified do nothing!!!

Classification Scheme

Rule 2

- When misclassifying a negative training example $\boldsymbol{x}$ $\in$Class 2 i.e. $\boldsymbol{w}^T\boldsymbol{x} \geq \theta$

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Properties of Winnow

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- If there are many irrelevant variables Winnow is better than the Perceptron.

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- Sensitive to the learning rate α.

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A variant of the Perceptron Algorithm

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Finally

It consists of the following steps

1. Initialize the weight vector $\boldsymbol{w}(0)$ in a random way.
2. Define a storage pocket vector $\boldsymbol{w}_s$ and a history counter h_s to zero for the same pocket vector.
3. At the i^{th} iteration step compute the update $\boldsymbol{w}(n+1)$ using the Perceptron rule.
4. Use the update weight to find the number h of samples correctly classified.
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6. Keep iterating to 3 until convergence.
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