Introduction to Artificial Intelligence

Belief Propagation and Junction Trees

Andres Mendez-Vazquez

March 8, 2019

Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
     - Tree Graphs
     - Junction Tree Formal Definition
     - Algorithm For Building Junction Trees
   - Example
     - Moralize the DAG
     - Triangulate
     - Listing of Cliques
   - Potential Function
   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
     - Lemma of Propagation of Information
     - Example
   - Now, the Full Propagation
     - Example of Propagation

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- The algorithm was first proposed by Judea Pearl in 1982, who formulated this algorithm on trees, and was later extended to polytrees.

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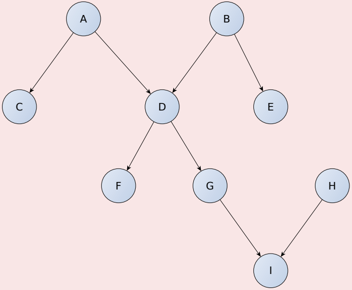

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Something Notable

- It has since been shown to be a useful approximate algorithm on general graphs.

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- The junction tree algorithm (also known as 'Clique Tree') is a method used in machine learning to extract marginalization in general graphs.

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- it entails performing belief propagation on a modified graph called a junction tree by cycle elimination

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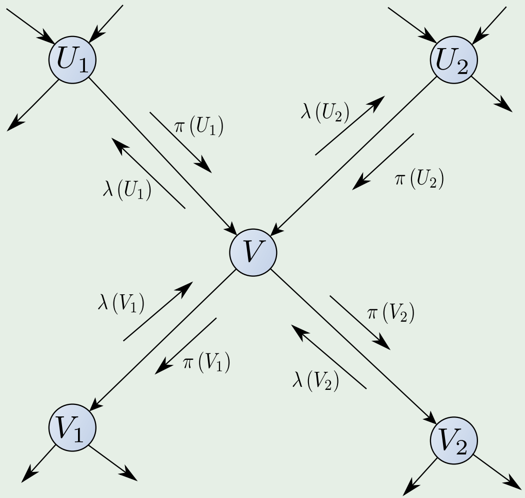
Example

The Message Passing Stuff



Thus

We can do the following

To pass information from below and from above to a certain node V.

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We call those messages

- π from above.
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Inference on Trees

Recall

A rooted tree is a DAG

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- Let (G, P) be a Bayesian network whose DAG is a tree.
- Let a be a set of values of a subset $A \subset V$.

- The general case can be inferred from it

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For simplicity

- Imagine that each node has two children.
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Then

Let D_X be the subset of A

- Containing all members that are in the subtree rooted at X
- Including X if $X \in A$

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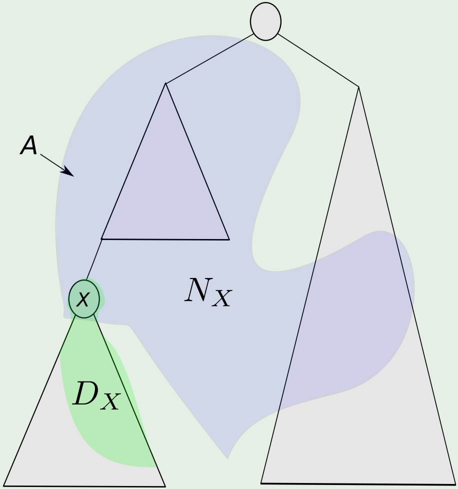
We have that $A = N_X \cup D_X$



Thus

We have for each value of x

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\begin{aligned}
P(x|A) = P(x|d_X, n_X) &= \frac{P(d_X, n_X|x)\,P(x)}{P(d_X, n_X)} \\
&= \frac{P(d_X|x, n_X)\,P(n_X|x)\,P(x)}{P(d_X, n_X)} \\
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&= \frac{P(d_X|x)\,P(x|n_X)\,P(n_X)}{P(d_X, n_X)} \quad \text{Here because } d\text{-speration if } X \notin A \\
&= \frac{P(d_X|x)\,P(x|n_X)\,P(n_X)}{P(d_X|n_X)\,P(n_X)}
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Note: You need to prove when $X \in A$

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$$P\left(x|A\right) = \frac{P\left(d_X|x\right)P\left(x|n_X\right)}{P\left(d_X|n_X\right)}$$
$$= \beta P\left(d_X|x\right)P\left(x|n_X\right)$$

where β, the normalizing factor, is a constant not depending on x.

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Now, we develop the messages

We want

- $\lambda(x) \simeq P(d_X|x)$
- $\pi(x) \simeq P(x|n_X)$
  - Where $\simeq$ means "proportional to"

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Meaning

$\pi(x)$ may not be equal to $P(x|n_X)$, but $\pi(x) = k \times P(x|n_X)$.

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$$P(x|a) = \alpha\lambda(x)\pi(x)$$

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Case 1: $X \in A$ and $X \in D_X$

Given any $X = \hat{x}$, we have that for $P(d_X|x) = 0$ for $x \neq \hat{x}$

- $\lambda(x) = 0$ for $x \neq \hat{x}$

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Thus, to achieve proportionality, we can set

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Case 2: $X \notin A$ and X is a leaf

Then, $d_X = \emptyset$ and

$$P(d_X|x) = P(\emptyset|x) = 1 \text{ for all values of } x$$

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Case 3: $X \notin A$ and X is a non-leaf

Let Y be X's left child, W be X's right child.

Since $X \notin A$

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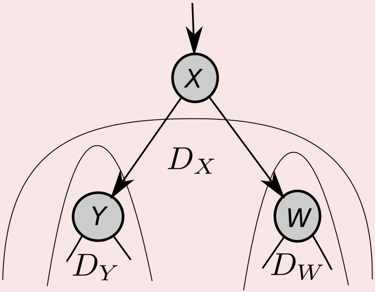

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&= \sum_y P\left(d_Y, y|x\right)\sum_w P\left(d_W, w|x\right)\\
&= \sum_y P\left(y|x\right)P\left(d_Y|y\right)\sum_w P\left(w|x\right)P\left(d_W|w\right)\\
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Thus, we can get proportionality by defining for all values of x

- $\lambda_Y\left(x\right) = \sum_y P\left(y|x\right) \lambda\left(y\right)$
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&= \sum_y P\left(y|x\right)P\left(d_Y|y\right)\sum_w P\left(w|x\right)P\left(d_W|w\right)\\
&\simeq \sum_y P\left(y|x\right)\lambda\left(y\right)\sum_w P\left(w|x\right)\lambda\left(w\right)
\end{aligned}
$$

Thus, we can get proportionality by defining for all values of x

- $\lambda_Y\left(x\right) = \sum_y P\left(y|x\right)\lambda\left(y\right)$
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We have then

$$\begin{aligned} P\left(d_X|x\right) &= P\left(d_Y, d_W|x\right) \\ &= P\left(d_Y|x\right) P\left(d_W|x\right) \text{ Because the d-separation at } X \\ &= \sum_y P\left(d_Y, y|x\right) \sum_w P\left(d_W, w|x\right) \\ &= \sum_y P\left(y|x\right) P\left(d_Y|y\right) \sum_w P\left(w|x\right) P\left(d_W|w\right) \\ &\simeq \sum_y P\left(y|x\right) \lambda\left(y\right) \sum_w P\left(w|x\right) \lambda\left(w\right) \end{aligned}$$

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Now

Case 2: $X \notin A$ and X is the root

In this specific case $n_X = \emptyset$ or the empty set of random variables.

Then

$$P(x|n_X) = P(x|\emptyset) = P(x) \text{ for all values of } x$$

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Case 3: $X \notin A$ and X is not the root

Without loss of generality assume X is Z's right child and T is the Z's left child

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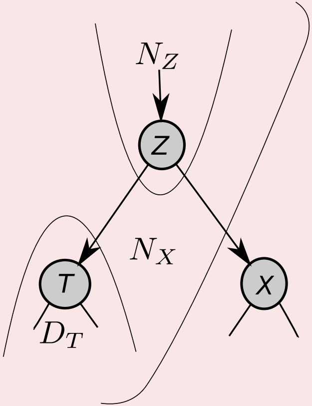

Then

We have

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&= \sum_z P\left(x|z\right) \frac{P\left(d_T|z\right) P\left(z|n_Z\right) P(n_Z)}{P\left(n_Z, d_T\right)} \text{ Again the d-separation for } z
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P(x|n_X) &= \sum_z P(x|z)\, P(z|n_X) \\
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Last Step

We have

$$P\left(x|n_X\right) = \sum_z P\left(x|z\right)\frac{P\left(z|n_Z\right)P\left(n_Z\right)P\left(d_T|z\right)}{P\left(n_Z, d_T\right)}$$
$$= \gamma\sum_z P\left(x|z\right)\pi\left(z\right)\lambda_T\left(z\right)$$

where $\gamma = \frac{P(n_Z)}{P(n_Z, d_T)}$

Thus, we can achieve proportionality by

$$\pi_X\left(z\right) = \pi\left(z\right)\lambda_T\left(z\right)$$

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Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
     - Tree Graphs
     - Junction Tree Formal Definition
     - Algorithm For Building Junction Trees
   - Example
     - Moralize the DAG
     - Triangulate
     - Listing of Cliques
   - Potential Function
   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
     - Lemma of Propagation of Information
     - Example
   - Now, the Full Propagation
     - Example of Propagation

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How do we implement this?

We require the following functions

- initial_tree
- update-tree

intial_tree has the following input and outputs

Input: $((G,P),$ A, a, $P(x|a))$

Output: After this call A and a are both empty making $P(x|a)$ the prior probability of x.

Then each time a variable V is instantiated for $\hat{v}$ the routine update-tree is called

Input: $((G,P),$ A, a, V, $\hat{v}$, $P(x|a))$

Output: After this call V has been added to A, $\hat{v}$ has been added to a and for every value of x, $P(x|a)$ has been updated to be the conditional probability of x given the new a.

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Problem

Given a Bayesian network whose DAG is a tree, determine the probabilities of the values of each node conditional on specified values of the nodes in some subset.

Input

Bayesian network (G, P) whose DAG is a tree, where $G = (V, E)$, and a set of values a of a subset A $\subseteq$ V.

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The Bayesian network (G, P) updated according to the values in a. The λ and π values and messages and $P(x|a)$ for each X$\in$V are considered part of the network.

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Initializing the tree

void initial_tree

input: **(Bayesian-network& $(\mathbb{G}, P)$ where $\mathbb{G} = (V, E)$, set-of-variables& A, set-of-variable-values& a)**

1. **A** $= \emptyset$
2. **a** $= \emptyset$
3. for (each X$\in$V)
4. for (each value x of X)
5. $\lambda(x) = 1$ // Compute λ values.
6. for (the parent Z of X) // Does nothing if X is the a root.
7. for (each value z of Z)
8. $\lambda_X(z) = 1$ // Compute λ messages.
9. for (each value r of the root R)
10. $P(r|\text{a}) = P(r)$ // Compute $P(r|\text{a})$.
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Updating the tree

void update_tree

Input: (Bayesian-network& $(\mathbb{G}, P)$ where $\mathbb{G} = (V, E)$, set-of-variables& A, set-of-variable-values& a, variable V , variable-value $\hat{v}$)

1. A $=$ A$\cup \{V\}$, a$=$ a$\cup \{\hat{v}\}$, $\lambda(\hat{v}) = 1$, $\pi(\hat{v}) = 1$, $P(\hat{v}|\text{a}) = 1$ // Add V to A and instantiate V to $\hat{v}$
2. a $= \emptyset$
3. for (each value of $v \neq \hat{v}$)
4. $\lambda(v) = 0$, $\pi(v) = 0$, $P(v|\text{a}) = 0$
5. if (V is not the root && V 's parent $Z \notin$A)
6. send_λ_msg(V, Z)
7. for (each child X of V such that $X \notin$A))
8. send_π_msg(V, X)

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1. A $=$ A$\cup\{V\}$, a$=$ a$\cup\{\hat{v}\}$, $\lambda\left(\hat{v}\right) = 1$, $\pi\left(\hat{v}\right) = 1$, $P(\hat{v}|$a$) = 1$ // Add V to A and instantiate V to $\hat{v}$
2. a $= \emptyset$
3. for (each value of $v \neq \hat{v}$)
4. $\quad\quad \lambda\left(v\right) = 0$, $\pi\left(v\right) = 0$, $P(v|$a$) = 0$
5. if (V is not the root && V 's parent $Z \notin$A)
6. $\quad\quad$ send_λ_msg(V, Z)
7. for (each child X of V such that $X \notin$A))
8. $\quad\quad$ send_π_msg(V, X)

Updating the tree

void update_tree

Input: (Bayesian-network& $(\mathbb{G}, P)$ where $\mathbb{G} = (V, E)$, set-of-variables& A, set-of-variable-values& a, variable V , variable-value $\hat{v}$)

1. A $=$ A$\cup\{V\}$, a$=$ a$\cup\{\hat{v}\}$, $\lambda(\hat{v}) = 1$, $\pi(\hat{v}) = 1$, $P(\hat{v}|$a$) = 1$ // Add V to A and instantiate V to $\hat{v}$
2. a $= \emptyset$
3. for (each value of $v \neq \hat{v}$)
4. $\quad\quad \lambda(v) = 0$, $\pi(v) = 0$, $P(v|$a$) = 0$
5. if (V is not the root && V 's parent $Z \notin$A)
6. $\quad\quad$ send_λ_msg(V, Z)
7. for (each child X of V such that $X \notin$A))
8. $\quad\quad$ send_π_msg(V, X)

Updating the tree

void update_tree

Input: (Bayesian-network& $(\mathbb{G}, P)$ where $\mathbb{G} = (V, E)$, set-of-variables& A, set-of-variable-values& a, variable V , variable-value $\hat{v}$)

1. A $=$ A$\cup \{V\}$, a$=$ a$\cup \{\hat{v}\}$, $\lambda(\hat{v}) = 1$, $\pi(\hat{v}) = 1$, $P(\hat{v}|$a$) = 1$ // Add V to A and instantiate V to $\hat{v}$
2. a $= \emptyset$
3. for (each value of $v \neq \hat{v}$)
4. $\quad$ $\lambda(v) = 0$, $\pi(v) = 0$, $P(v|$a$) = 0$
5. if (V is not the root && V 's parent $Z \notin$A)
6. $\quad$ send_λ_msg(V, Z)
7. for (each child X of V such that $X \notin$A))
8. $\quad$ send_π_msg(V, X)

Updating the tree

void update_tree

Input: (Bayesian-network& $(\mathbb{G}, P)$ where $\mathbb{G} = (V, E)$, set-of-variables& A, set-of-variable-values& a, variable V , variable-value $\hat{v}$)

1. A $=$ A$\cup \{V\}$, a$=$ a$\cup \{\hat{v}\}$, $\lambda\left(\hat{v}\right) = 1$, $\pi\left(\hat{v}\right) = 1$, $P(\hat{v}|$a$) = 1$ // Add V to A and instantiate V to $\hat{v}$
2. a $= \emptyset$
3. for (each value of $v \neq \hat{v}$)
4. $\quad\lambda\left(v\right) = 0$, $\pi\left(v\right) = 0$, $P(v|$a$) = 0$
5. if (V is not the root && V 's parent $Z \notin$A)
6. $\quad$send_λ_msg(V, Z)
7. for (each child X of V such that $X \notin$A))
8. $\quad$send_π_msg(V, X)

Sending the λ message

void send_λ_msg(node Y , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

1. for (each value of x)
2. $\quad \lambda_Y(x) = \sum_y P(y|x)\,\lambda(y)$ // Y sends X a λ message
3. $\quad \lambda(x) = \prod_{U \in CH_X} \lambda_U(x)$ // Compute $X's$ λ values
4. $\quad P(x|\text{a}) = \alpha\lambda(x)\,\pi(x)$ // Compute $P(x|\text{a})$
5. normalize $P(x|\text{a})$
6. if (X is not the root and $X's$ parent $Z \notin$A)
7. $\quad$ send_λ_msg(X, Z)
8. for (each child W of X such that $W \neq Y$ and $W \in$A))
9. $\quad$ send_π_msg(X, W)

Sending the λ message

void send_λ_msg(node Y , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

1. for (each value of x)
2. $\quad\lambda_Y(x) = \sum_y P(y|x)\lambda(y)$ // Y sends X a λ message
3. $\quad\lambda(x) = \prod_{U \in CH_X} \lambda_U(x)$ // Compute $X's$ λ values
4. $\quad P(x|\mathrm{a}) = \alpha\lambda(x)\pi(x)$ // Compute $P(x|\mathrm{a})$
5. normalize $P(x|\mathrm{a})$
6. if (X is not the root and $X's$ parent $Z \notin$A)
7. $\quad$send_λ_msg(X, Z)
8. for (each child W of X such that $W \neq Y$ and $W \in$A))
9. $\quad$send_π_msg(X, W)

Sending the λ message

void send_λ_msg(node Y , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

1. for (each value of x)
2. $\quad \lambda_Y(x) = \sum_y P(y|x)\lambda(y)$ // Y sends X a λ message
3. $\quad \lambda(x) = \prod_{U \in CH_X} \lambda_U(x)$ // Compute $X's$ λ values
4. $\quad P(x|\text{a}) = \alpha\lambda(x)\pi(x)$ // Compute $P(x|\text{a})$
5. normalize $P(x|\text{a})$
6. if (X is not the root and $X's$ parent $Z \notin$A)
7. $\quad$ send_λ_msg(X, Z)
8. for (each child W of X such that $W \neq Y$ and $W \in$A))
9. $\quad$ send_π_msg(X, W)

Sending the λ message

void send_λ_msg(node Y , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

for (each value of x)
    $\lambda_Y(x) = \sum_y P(y|x)\lambda(y)$ // Y sends X a λ message
    $\lambda(x) = \prod_{U \in CH_X} \lambda_U(x)$ // Compute $X's$ λ values
    $P(x|\text{a}) = \alpha\lambda(x)\pi(x)$ // Compute $P(x|\text{a})$
normalize $P(x|\text{a})$
if (X is not the root and $X's$ parent $Z \notin$A)
    send_λ_msg(X, Z)
for (each child W of X such that $W \neq Y$ and $W \in$A))
    send_π_msg(X, W)

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Sending the π message

void send_π_msg(node Z , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

1. for (each value of z)
2. $\quad \pi_X(z) = \pi(z) \prod_{Y \in CH_Z - \{X\}} \lambda_Y(z)$ // Z sends X a π message
3. for (each value of x)
4. $\quad \pi(x) = \sum_z P(x|z) \pi_X(z)$ // Compute$X's$ π values
5. $\quad P(x|a) = \alpha \lambda(x) \pi(x)$ // Compute $P(x|$a)
6. normalize $P(x|a)$
7. for (each child Y of X such that $Y \notin$A))
8. $\quad$ send_π_msg(X, Y)

Sending the π message

void send_π_msg(node Z , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

1. for (each value of z)
2. $\quad\pi_X(z) = \pi(z) \prod_{Y \in CH_Z - \{X\}} \lambda_Y(z)$ // Z sends X a π message
3. for (each value of x)
4. $\quad\pi(x) = \sum_z P(x|z)\, \pi_X(z)$ // Compute$X's$ π values
5. $\quad P(x|a) = \alpha\lambda(x)\, \pi(x)$ // Compute $P(x|$a)
6. normalize $P(x|a)$
7. for (each child Y of X such that $Y \notin$A))
8. $\quad$send_π_msg(X, Y)

Sending the π message

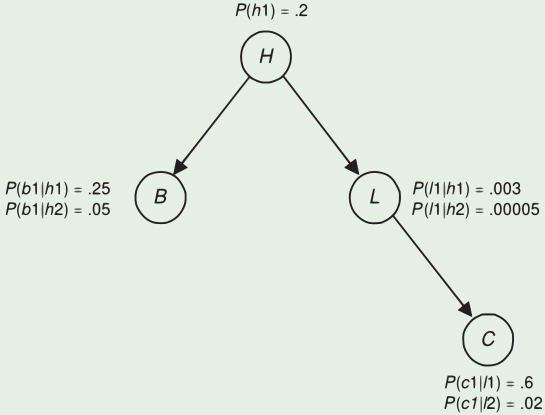
void send_π_msg(node Z , node X)

Note: For simplicity $(\mathbb{G}, P)$ is not shown as input.

1. for (each value of z)
2. $\quad \pi_X(z) = \pi(z) \prod_{Y \in CH_Z - \{X\}} \lambda_Y(z)$ // Z sends X a π message
3. for (each value of x)
4. $\quad \pi(x) = \sum_z P(x|z)\, \pi_X(z)$ // Compute$X's$ π values
5. $\quad P(x|a) = \alpha\lambda(x)\, \pi(x)$ // Compute $P(x|$a)
6. normalize $P(x|a)$
7. for (each child Y of X such that $Y \notin$A))
8. $\quad$ send_π_msg(X, Y)

Example of Tree Initialization

We have then



Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A$=\emptyset$, a$=\emptyset$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1;\lambda(h2) = 1;$
- $\lambda(b1) = 1;\lambda(b2) = 1;$
- $\lambda(l1) = 1;\lambda(l2) = 1;$
- $\lambda(c1) = 1;\lambda(c2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1; \lambda(h2) = 1;$
- $\lambda(b1) = 1; \lambda(b2) = 1;$
- $\lambda(l1) = 1; \lambda(l2) = 1;$
- $\lambda(c1) = 1; \lambda(c2) = 1;$

Compute λ_{μ} messages

- $\lambda_B(h1) = 1; \lambda_B(h2) = 1;$
- $\lambda_L(h1) = 1; \lambda_L(h2) = 1;$
- $\lambda_C(l1) = 1; \lambda_C(l2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1;\lambda(h2) = 1;$
- $\lambda(b1) = 1; \lambda(b2) = 1;$
- $\lambda(l1) = 1; \lambda(l2) = 1;$
- $\lambda(c1) = 1; \lambda(c2) = 1;$

Compute λ_{s} messages

- $\lambda_B(h1) = 1; \lambda_B(h2) = 1;$
- $\lambda_L(h1) = 1; \lambda_L(h2) = 1;$
- $\lambda_C(l1) = 1; \lambda_C(l2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1;\lambda(h2) = 1;$
- $\lambda(b1) = 1; \lambda(b2) = 1;$
- $\lambda(l1) = 1; \lambda(l2) = 1;$
- $\lambda(c1) = 1; \lambda(c2) = 1;$

Compute λ_v messages

- $\lambda_B(h1) = 1; \lambda_B(h2) = 1;$
- $\lambda_L(h1) = 1; \lambda_L(h2) = 1;$
- $\lambda_C(l1) = 1; \lambda_C(l2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1; \lambda(h2) = 1;$
- $\lambda(b1) = 1; \lambda(b2) = 1;$
- $\lambda(l1) = 1; \lambda(l2) = 1;$
- $\lambda(c1) = 1; \lambda(c2) = 1;$

Compute λ_v messages

- $\lambda_B(h1) = 1; \lambda_B(h2) = 1;$
- $\lambda_L(h1) = 1; \lambda_L(h2) = 1;$
- $\lambda_C(l1) = 1; \lambda_C(l2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1; \lambda(h2) = 1;$
- $\lambda(b1) = 1; \lambda(b2) = 1;$
- $\lambda(l1) = 1; \lambda(l2) = 1;$
- $\lambda(c1) = 1; \lambda(c2) = 1;$

Compute λ_v messages

- $\lambda_B(h1) = 1; \lambda_B(h2) = 1;$
- $\lambda_L(h1) = 1; \lambda_L(h2) = 1;$
- $\lambda_C(l1) = 1; \lambda_C(l2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

We have then

- A=$\emptyset$, a=$\emptyset$

Compute λ values

- $\lambda(h1) = 1;\lambda(h2) = 1;$
- $\lambda(b1) = 1; \lambda(b2) = 1;$
- $\lambda(l1) = 1; \lambda(l2) = 1;$
- $\lambda(c1) = 1; \lambda(c2) = 1;$

Compute λ_v messages

- $\lambda_B(h1) = 1; \lambda_B(h2) = 1;$
- $\lambda_L(h1) = 1; \lambda_L(h2) = 1;$
- $\lambda_C(l1) = 1; \lambda_C(l2) = 1;$

Calling initial_tree(($\mathbb{G}, P$), A, a)

Compute $P(h|\emptyset)$

- $P(h1|\emptyset) = P(h1) = 0.2$
- $P(h2|\emptyset) = P(h2) = 0.8$

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Calling initial_tree(($\mathbb{G}, P$), A, a)

Compute $P\left(h|\emptyset\right)$

- $P(h1|\emptyset) = P(h1) = 0.2$
- $P(h2|\emptyset) = P(h2) = 0.8$

Compute H's π values

- $\pi(h1) = P(h1) = 0.2$
- $\pi(h2) = P(h2) = 0.8$

- send_π_msg(H, L)

Calling initial_tree(($\mathbb{G}, P$), A, a)

Compute $P(h|\emptyset)$

- $P(h1|\emptyset) = P(h1) = 0.2$
- $P(h2|\emptyset) = P(h2) = 0.8$

Compute H's π values

- $\pi(h1) = P(h1) = 0.2$
- $\pi(h2) = P(h2) = 0.8$

Send messages

- send_π_msg(H, B)
- send_π_msg(H, L)

Calling initial_tree(($\mathbb{G}, P$), A, a)

Compute $P(h|\emptyset)$

- $P(h1|\emptyset) = P(h1) = 0.2$
- $P(h2|\emptyset) = P(h2) = 0.8$

Compute H's π values

- $\pi(h1) = P(h1) = 0.2$
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Send messages

- send_π_msg(H, B)
- send_π_msg(H, L)

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Calling initial_tree(($\mathbb{G}, P$), A, a)

Compute $P(h|\emptyset)$

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Compute H's π values

- $\pi(h1) = P(h1) = 0.2$
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Send messages

- send_π_msg(H, B)
- send_π_msg(H, L)

Calling initial_tree(($\mathbb{G}, P$), A, a)

Compute $P(h|\emptyset)$

- $P(h1|\emptyset) = P(h1) = 0.2$
- $P(h2|\emptyset) = P(h2) = 0.8$

Compute H's π values

- $\pi(h1) = P(h1) = 0.2$
- $\pi(h2) = P(h2) = 0.8$

Send messages

- send_π_msg(H, B)
- send_π_msg(H, L)

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The call send_π_msg(H, B)

H sends B a π message

- $\pi_B(h1) = \pi(h1)\lambda_L(h1) = 0.2 \times 1 = 0.2$
- $\pi_B(h2) = \pi(h2)\lambda_L(h2) = 0.8 \times 1 = 0.8$

The call send_π_msg(H, B)

H sends B a π message

- $\pi_B(h1) = \pi(h1)\lambda_L(h1) = 0.2 \times 1 = 0.2$
- $\pi_B(h2) = \pi(h2)\lambda_L(h2) = 0.8 \times 1 = 0.8$

Compute B's π values

$$\begin{aligned}
\pi(b1) &= P(b1|h1)\,\pi_B(h1) + P(b1|h2)\,\pi_B(h2) \\
&= (0.25)(0.2) + (0.05)(0.8) = 0.09 \\
\pi(b2) &= P(b2|h1)\,\pi_B(h1) + P(b2|h2)\,\pi_B(h2) \\
&= (0.75)(0.2) + (0.95)(0.8) = 0.91
\end{aligned}$$

The call send_π_msg(H, B)

H sends B a π message

- $\pi_B(h1) = \pi(h1)\lambda_L(h1) = 0.2 \times 1 = 0.2$
- $\pi_B(h2) = \pi(h2)\lambda_L(h2) = 0.8 \times 1 = 0.8$

Compute B's π values

$$\begin{aligned}
\pi(b1) &= P(b1|h1)\,\pi_B(h1) + P(b1|h2)\,\pi_B(h2) \\
&= (0.25)(0.2) + (0.05)(0.8) = 0.09 \\
\pi(b2) &= P(b2|h1)\,\pi_B(h1) + P(b2|h2)\,\pi_B(h2) \\
&= (0.75)(0.2) + (0.95)(0.8) = 0.91
\end{aligned}$$

The call send_π_msg(H, B)

H sends B a π message

- $\pi_B(h1) = \pi(h1)\lambda_L(h1) = 0.2 \times 1 = 0.2$
- $\pi_B(h2) = \pi(h2)\lambda_L(h2) = 0.8 \times 1 = 0.8$

Compute B's π values

$$\begin{aligned}\pi(b1) &= P(b1|h1)\,\pi_B(h1) + P(b1|h2)\,\pi_B(h2)\\ &= (0.25)(0.2) + (0.05)(0.8) = 0.09\end{aligned}$$

$$\begin{aligned}\pi(b2) &= P(b2|h1)\,\pi_B(h1) + P(b2|h2)\,\pi_B(h2)\\ &= (0.75)(0.2) + (0.95)(0.8) = 0.91\end{aligned}$$

The call send_π_msg(H, B)

H sends B a π message

- $\pi_B(h1) = \pi(h1)\lambda_L(h1) = 0.2 \times 1 = 0.2$
- $\pi_B(h2) = \pi(h2)\lambda_L(h2) = 0.8 \times 1 = 0.8$

Compute B's π values

$$
\begin{aligned}
\pi\left(b1\right) &= P\left(b1|h1\right)\pi_B\left(h1\right) + P\left(b1|h2\right)\pi_B\left(h2\right)\\
&= \left(0.25\right)\left(0.2\right) + \left(0.05\right)\left(0.8\right) = 0.09\\
\pi\left(b2\right) &= P\left(b2|h1\right)\pi_B\left(h1\right) + P\left(b2|h2\right)\pi_B\left(h2\right)\\
&= \left(0.75\right)\left(0.2\right) + \left(0.95\right)\left(0.8\right) = 0.91
\end{aligned}
$$

The call send_π_msg(H, B)

H sends B a π message

- $\pi_B(h1) = \pi(h1)\lambda_L(h1) = 0.2 \times 1 = 0.2$
- $\pi_B(h2) = \pi(h2)\lambda_L(h2) = 0.8 \times 1 = 0.8$

Compute B's π values

$$\begin{aligned}
\pi\left(b1\right) &= P\left(b1|h1\right)\pi_B\left(h1\right) + P\left(b1|h2\right)\pi_B\left(h2\right)\\
&= \left(0.25\right)\left(0.2\right) + \left(0.05\right)\left(0.8\right) = 0.09\\
\pi\left(b2\right) &= P\left(b2|h1\right)\pi_B\left(h1\right) + P\left(b2|h2\right)\pi_B\left(h2\right)\\
&= \left(0.75\right)\left(0.2\right) + \left(0.95\right)\left(0.8\right) = 0.91
\end{aligned}$$

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The call send_π_msg(H, B)

Compute $P(b|\emptyset)$

- $P(b1|\emptyset) = \alpha\lambda(b1)\pi(b1) = \alpha(1)(0.09) = 0.09\alpha$
- $P(b2|\emptyset) = \alpha\lambda(b2)\pi(b2) = \alpha(1)(0.91) = 0.91\alpha$

The call send_π_msg(H, B)

Compute $P(b|\emptyset)$

- $P(b1|\emptyset) = \alpha\lambda(b1)\,\pi(b1) = \alpha(1)(0.09) = 0.09\alpha$
- $P(b2|\emptyset) = \alpha\lambda(b2)\,\pi(b2) = \alpha(1)(0.91) = 0.91\alpha$

Then, normalize

$$P(b1|\emptyset) = \frac{0.09\alpha}{0.09\alpha + 0.91\alpha} = 0.09$$

$$P(b2|\emptyset) = \frac{0.91\alpha}{0.09\alpha + 0.91\alpha} = 0.91$$

The call send_π_msg(H, B)

Compute $P(b|\emptyset)$

- $P(b1|\emptyset) = \alpha\lambda(b1)\pi(b1) = \alpha(1)(0.09) = 0.09\alpha$
- $P(b2|\emptyset) = \alpha\lambda(b2)\pi(b2) = \alpha(1)(0.91) = 0.91\alpha$

Then, normalize

$$P(b1|\emptyset) = \frac{0.09\alpha}{0.09\alpha + 0.91\alpha} = 0.09$$

$$P(b2|\emptyset) = \frac{0.91\alpha}{0.09\alpha + 0.91\alpha} = 0.91$$

The call send_π_msg(H, B)

Compute $P\left(b|\emptyset\right)$

- $P\left(b1|\emptyset\right) = \alpha\lambda\left(b1\right)\pi\left(b1\right) = \alpha\left(1\right)\left(0.09\right) = 0.09\alpha$
- $P\left(b2|\emptyset\right) = \alpha\lambda\left(b2\right)\pi\left(b2\right) = \alpha\left(1\right)\left(0.91\right) = 0.91\alpha$

Then, normalize

$$P\left(b1|\emptyset\right) = \frac{0.09\alpha}{0.09\alpha + 0.91\alpha} = 0.09$$
$$P\left(b2|\emptyset\right) = \frac{0.91\alpha}{0.09\alpha + 0.91\alpha} = 0.91$$

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Send the call send_π_msg(H, L)

H sends L a π message

- $\pi_L\,(h1) = \pi\,(h1)\,\lambda_B\,(h1) = (0.2)\,(1) = 0.2$
- $\pi_L\,(h2) = \pi\,(h2)\,\lambda_B\,(h2) = (0.8)\,(1) = 0.8$

Send the call send_π_msg(H, L)

H sends L a π message

- $\pi_L(h1) = \pi(h1)\,\lambda_B(h1) = (0.2)(1) = 0.2$
- $\pi_L(h2) = \pi(h2)\,\lambda_B(h2) = (0.8)(1) = 0.8$

Compute $L's$ π values

$$
\begin{aligned}
\pi(l1) &= P(l1|h1)\,\pi_L(h1) + P(l1|h2)\,\pi_L(h2) \\
&= (0.003)(0.2) + (0.00005)(0.8) = 0.00064 \\
\pi(l2) &= P(l2|h1)\,\pi_B(h1) + P(l2|h2)\,\pi_B(h2) \\
&= (0.997)(0.2) + (0.99995)(0.8) = 0.99936
\end{aligned}
$$

Send the call send_π_msg(H, L)

H sends L a π message

- $\pi_L(h1) = \pi(h1)\lambda_B(h1) = (0.2)(1) = 0.2$
- $\pi_L(h2) = \pi(h2)\lambda_B(h2) = (0.8)(1) = 0.8$

Compute $L's$ π values

$$\begin{aligned}\pi(l1) &= P(l1|h1)\pi_L(h1) + P(l1|h2)\pi_L(h2)\\ &= (0.003)(0.2) + (0.00005)(0.8) = 0.00064\end{aligned}$$

$$\begin{aligned}\pi(l2) &= P(l2|h1)\pi_B(h1) + P(l2|h2)\pi_B(h2)\\ &= (0.997)(0.2) + (0.99995)(0.8) = 0.99936\end{aligned}$$

Compute $P(l|\emptyset)$

- $P(l1|\emptyset) = \alpha\lambda(l1)\pi(l1) = \alpha(1)(0.00064) = 0.00064\alpha$
- $P(l2|\emptyset) = \alpha\lambda(l2)\pi(l2) = \alpha(1)(0.99936) = 0.99936\alpha$

Send the call send_π_msg(H, L)

H sends L a π message

- $\pi_L(h1) = \pi(h1)\lambda_B(h1) = (0.2)(1) = 0.2$
- $\pi_L(h2) = \pi(h2)\lambda_B(h2) = (0.8)(1) = 0.8$

Compute $L's$ π values

$$\begin{aligned}
\pi(l1) &= P(l1|h1)\pi_L(h1) + P(l1|h2)\pi_L(h2) \\
&= (0.003)(0.2) + (0.00005)(0.8) = 0.00064 \\
\pi(l2) &= P(l2|h1)\pi_B(h1) + P(l2|h2)\pi_B(h2) \\
&= (0.997)(0.2) + (0.99995)(0.8) = 0.99936
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$$
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\pi\left(c1\right) &= P\left(c1|l1\right)\pi_C\left(l1\right) + P\left(c1|l2\right)\pi_C\left(l2\right)\\
&= \left(0.6\right)\left(0.00064\right) + \left(0.02\right)\left(0.99936\right) = 0.02037\\
\pi\left(c2\right) &= P\left(c2|l1\right)\pi_C\left(h1\right) + P\left(c2|l2\right)\pi_C\left(l2\right)\\
&= \left(0.4\right)\left(0.00064\right) + \left(0.98\right)\left(0.99936\right) = 0.97963
\end{aligned}
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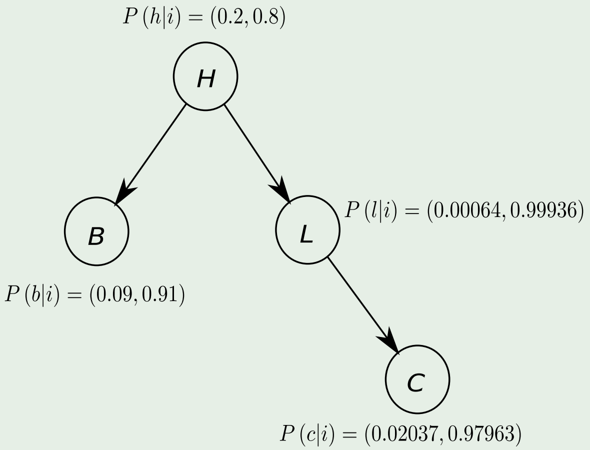
Final Graph

We have then



For the Generalization Please look at...

Look at pages 123 - 156 at

Richard E. Neapolitan. 2003. Learning Bayesian Networks. Prentice-Hall, Inc

History

Invented in 1988

Invented by Lauritzen and Spiegelhalter, 1988

Something Notable

The general idea is that the propagation of evidence through the network can be carried out more efficiently by representing the joint probability distribution on an undirected graph called the Junction tree (or Join tree).

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More in the Intuition

High-level Intuition

Computing marginals is straightforward in a tree structure.

Junction Tree Characteristics

The junction tree has the following characteristics

- It is an undirected tree
- Its nodes are clusters of variables (i.e. from the original BN)
- Given two clusters, C_1 and C_2, every node on the path between them contains their intersection $C_1 \cap C_2$

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A Separator, S, is associated with each edge and contains the variables in the intersection between neighboring nodes

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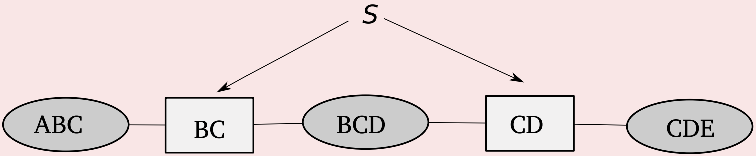

Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
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3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
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Simplicial Node

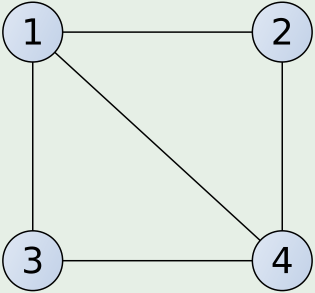
Simplicial Node

In a graph G, a vertex v is called **simplicial** if and only if the subgraph of G induced by the vertex set $\{v\} \cup N(v)$ is a clique.

- $N(v)$ is the neighbor of v in the Graph.

Example

Vertex 3 is simplicial, while 4 is not



Perfect Elimination Ordering

Definition

A graph G on n vertices is said to have a **perfect elimination ordering** if and only if there is an ordering $\{v_1, ..., v_n\}$ of G's vertices, such that each v_i is simplicial in the subgraph induced by the vertices $\{v_1, ..., v_i\}$.

Clearly

This is a way to collapse seto of vertices

- Into a single node... for graph simplification... using the cliques....

Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
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     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
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Chordal Graph

Definition

A Chordal Graph is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

Definition

For any two vertices $x, y \in G$ such that $(x, y) \in E$, a $x - y$ separator is a set $S \subset V$ such that the graph $G - S$ has at least two disjoint connected components, one of which contains x and another of which contains y.

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Chordal Graph

Theorem

For a graph G on n vertices, the following conditions are equivalent:

1. G has a perfect elimination ordering.
2. G is chordal.
3. If H is any induced subgraph of G and S is a vertex separator of H of minimal size, S's vertices induce a clique.

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1 Introduction
- What do we want?

2 Belief Propagation
- The Intuition
- Inference on Trees
  - The Messages
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3 Junction Trees
- The Junction Tree Concept
  - Chordal Graphs
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We have the the following Claims

1. A chordal graph with N vertices can have no more than N maximal cliques.
2. Given a chordal graph with $G = (V, E)$, where $|V| = N$, there exists an algorithm to find all the maximal cliques of G which takes no more than $O(N^4)$ time.

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Elimination Clique

Definition (Elimination Clique)

Given a chordal graph G, and an elimination ordering for G which does not add any edges.

- Suppose node i (Assuming a Labeling) is eliminated in some step of the elimination algorithm, then the clique consisting of the node i along with its neighbors during the elimination step (which must be fully connected since elimination does not add edges) is called an elimination clique.

Formally

Suppose node i is eliminated in the k^{th} step of the algorithm, and let $G^{(k)}$ be the graph just before the k^{th} elimination step. Then, the clique $C_i = \{i\} \cup N^{(k)}(i)$ where $N^{(k)}(i)$ is the neighbor of i in the Graph $G^{(k)}$.

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From this, we have

Theorem

Given a chordal graph and an elimination ordering which does not add any edges. Let $\mathcal{C}$ be the set of maximal cliques in the chordal graph, and let $\mathcal{C}_e = (\cup_{i \in V} C_i)$ be the set of elimination cliques obtained from this elimination ordering. Then, $\mathcal{C} \subseteq \mathcal{C}_e$. In other words, every maximal clique is also an elimination clique for this particular ordering.

Something Notable

The theorem proves the 2^{nd} claims given earlier. Firstly, it shows that a chordal graph cannot have more than N maximal cliques, since we have only N elimination cliques.

It is more

It gives us an efficient algorithm for finding these N maximal cliques.

- Simply go over each elimination clique and check whether it is maximal.

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Therefore

Even with a brute force approach

It will not take more than $|\mathcal{C}_e|^2 \times D = O\left(N^3\right)$ with $D = \max_{C \in \mathcal{C}} |C|$.

Because

Since both clique size and number of elimination cliques is bounded by N

Observation

The maximum clique problem, which is NP-hard on general graphs, is easy on chordal graphs.

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   - The Intuition
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     - The Messages
     - The Implementation
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   - The Junction Tree Concept
     - Chordal Graphs
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We have the following definitions

Definition

The following are equivalent to the statement "G is a tree"

1. G is a connected, acyclic graph over N nodes.
2. G is a connected graph over N nodes with $N-1$ edges.
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4. (**Important**) G is a graph over N nodes, such that for any 2 nodes i and j in G , with $i \neq j$, there is a unique path from i to j in G.

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3. Junction Trees
   - The Junction Tree Concept
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Definition

Junction Tree

Given a graph $G = (V, E)$, a graph $G' = (V', E')$ is said to be a Junction Tree for G, iff:

1. The nodes of G' are the maximal cliques of G (i.e. G' is a clique graph of G.)
2. G' is a tree.
3. Running Intersection Property / Junction Tree Property:
   1. For each $v \in V$, define G'_v to be the induced subgraph of G' consisting of exactly those nodes which correspond to maximal cliques of G that contain v. Then G'_v must be a connected graph.

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Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
     - Tree Graphs
     - Junction Tree Formal Definition
     - Algorithm For Building Junction Trees
   - Example
     - Moralize the DAG
     - Triangulate
     - Listing of Cliques
   - Potential Function
   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
     - Lemma of Propagation of Information
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   - Now, the Full Propagation
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Step 1

Given a DAG $G=(V,E)$ and $|V|=N$

Chordalize the graph using the elimination algorithm with an arbitrary elimination ordering, if required.

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Given a list of nodes:

1. Is the vertex simplicial? If it is not, make it simplicial.
2. If not remove it from the list.

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Another way

1. By the Moralization Procedure.
2. Triangulate the moral graph.

Moralization Procedure

1. Add edges between all pairs of nodes that have a common child.

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1. Add edges between all pairs of nodes that have a common child.
2. Make all edges in the graph undirected.

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Moralization Procedure

1. Add edges between all pairs of nodes that have a common child.
2. Make all edges in the graph undirected.

Triangulate the moral graph

An undirected graph is triangulated if every cycle of length greater than 3 possesses a chord.

Step 2

Find the maximal cliques in the chordal graph

List the N Cliques

- $(\{v_N\} \cup N(v_N)) \cap \{v_1, ..., v_N\}$
- $(\{v_{N-1}\} \cup N(v_{N-1})) \cap \{v_1, ..., v_{N-1}\}$
- $\cdots$
- $\{v_1\}$

Note: If the graph is Chordal this is not necessary because all the cliques are maximal.

Step 3

Compute the separator sets for each pair of maximal cliques and construct a weighted clique graph

For each pair of maximal cliques (C_i, C_j) in the graph

- We check whether they posses any common variables.

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If yes, we designate a separator set

Between these 2 cliques as $S_{ij} = C_i \cap C_j$.

- Nodes are the Cliques.
- Edges (C_i, C_j) are added with weight $|C_i \cap C_j|$

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Then, we compute these separators trees

We build a clique graph:

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Step 3

This step can be implemented quickly in practice using a hash table

Running Time: $O\left(\left|\mathcal{C}\right|^2 D\right) = O\left(N^2 D\right)$

Step 4

Compute a maximum-weight spanning tree on the weighted clique graph to obtain a junction tree

You can us for this the Kruskal and Prim for Maximum Weight Graph

We will give Kruskal's algorithm

For finding the maximum-weight spanning tree

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Step 4

Maximal Kruskal's algorithm

Initialize an edgeless graph $\mathcal{T}$ with nodes that are all the maximal cliques in our chordal graph.

Then

We will add edges to $\mathcal{T}$ until it becomes a junction tree.

Sort the m edges e_i in our clique graph from step 3 by weight w_i

We have for $e_1, e_2, ..., e_m$ with $w_1 \geq w_2 \geq \cdots \geq w_1$

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Step 4

For $i = 1, 2, ..., m$

1. Add edge e_i to $\mathcal{T}$ if it does not introduce a cycle.
2. If $|\mathcal{C}| - 1$ edges have been added, quit.

Running Time given that $|E| = O\left(|\mathcal{C}|^2\right)$

$$O\left(|\mathcal{C}|^2 \log |\mathcal{C}|^2\right) = O\left(|\mathcal{C}|^2 \log |\mathcal{C}|\right) = O\left(N^2 \log N\right)$$

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Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
     - Tree Graphs
     - Junction Tree Formal Definition
     - Algorithm For Building Junction Trees
   - Example
     - Moralize the DAG
     - Triangulate
     - Listing of Cliques
   - Potential Function
   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
     - Lemma of Propagation of Information
     - Example
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Outline

1 Introduction
- What do we want?

2 Belief Propagation
- The Intuition
- Inference on Trees
  - The Messages
  - The Implementation

3 Junction Trees
- The Junction Tree Concept
  - Chordal Graphs
  - Maximal Clique
  - Tree Graphs
  - Junction Tree Formal Definition
  - Algorithm For Building Junction Trees
- Example
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- The Junction Tree Inference Algorithms
- Propagating Information in a Junction Tree
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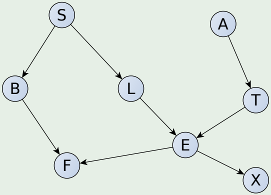
How do you build a Junction Tree?

Given a General DAG



Build a Chordal Graph

- Moral Graph – marry common parents and remove arrows.

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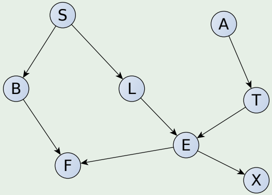

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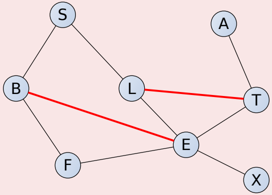

Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
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     - Junction Tree Formal Definition
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   - Example
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     - Triangulate
     - Listing of Cliques
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   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
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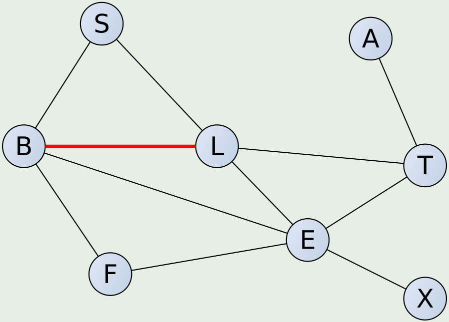

Outline

1. Introduction
   - What do we want?

2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation

3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
     - Tree Graphs
     - Junction Tree Formal Definition
     - Algorithm For Building Junction Trees
   - Example
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     - Triangulate
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   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
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     - Example
   - Now, the Full Propagation
     - Example of Propagation

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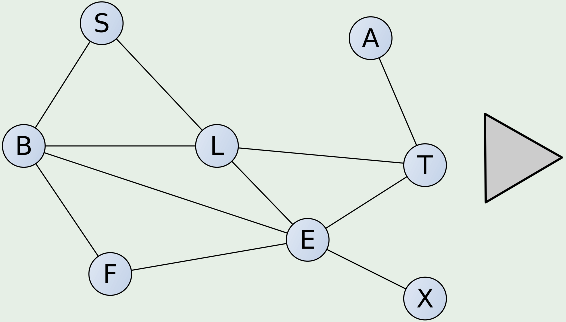
Listing of Cliques

Identify the Cliques

- A clique is a subset of nodes which is **complete** (i.e. there is an edge between every pair of nodes) and **maximal**.

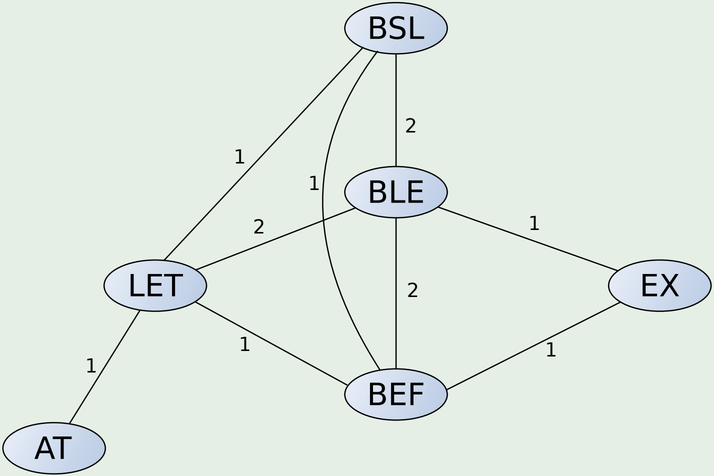


{B,S,L}
{B,L,E}
{B,E,F}
{L,E,T}
{A,T}
{E,X}

Build the Clique Graph

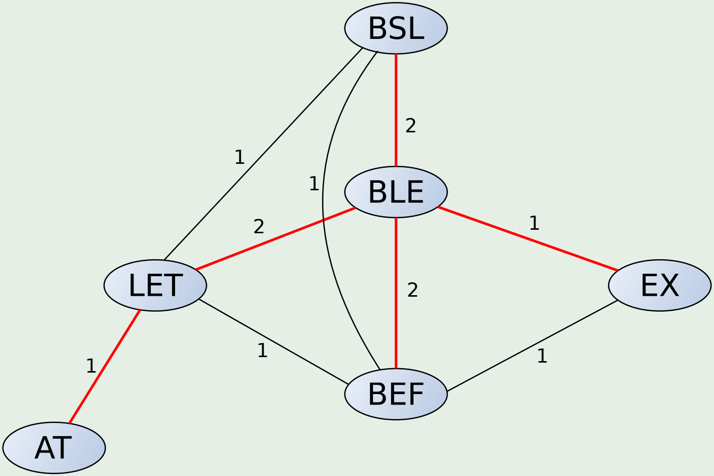
Clique Graph

- Add an edge between C_j and C_i with weight $|C_i \cap C_j| > 0$



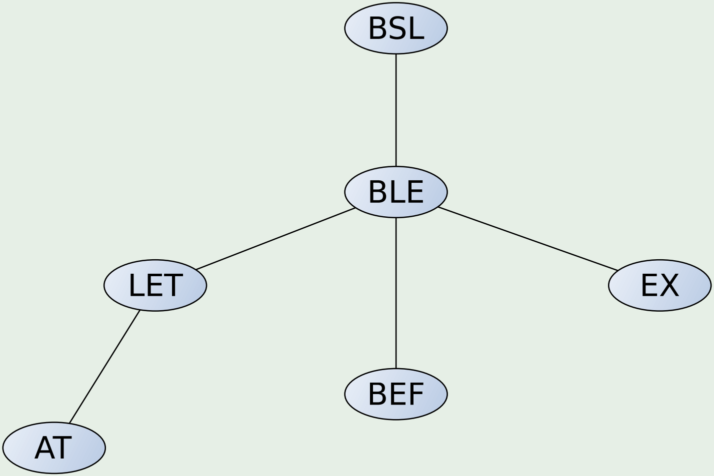
Getting The Junction Tree

Run the Maximum Kruskal's Algorithm



Getting The Junction Tree

Finally



Outline

1. Introduction
   - What do we want?
2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation
3. Junction Trees
   - The Junction Tree Concept
     - Chordal Graphs
     - Maximal Clique
     - Tree Graphs
     - Junction Tree Formal Definition
     - Algorithm For Building Junction Trees
   - Example
     - Moralize the DAG
     - Triangulate
     - Listing of Cliques
   - Potential Function
   - The Junction Tree Inference Algorithms
   - Propagating Information in a Junction Tree
     - Update
     - Lemma of Propagation of Information
     - Example
   - Now, the Full Propagation
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Potential as a product of probabilites

We can think on a clique as a place were the all the info is shared between variables

$$x_{c_1}, ..., x_{c_n}$$

Thus, all they are independent between them

$$P\left(x_{c_1}, ..., x_{c_n}\right) = \frac{1}{Z}\prod_{i=1}^{n}\varphi_C\left(x_{c_i}\right)$$

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Potential Representation for the Junction Tree

Then

- The joint probability distribution can now be represented in terms of potential functions, φ_C.
  - This is defined in each clique and each separator

The basic idea is to represent the joint probability distribution corresponding to any graph as a product of clique potentials

$$P\left(x_{c_1},\ldots,x_{c_n}\right)=\frac{1}{Z}\prod_{i=1}^{n}\varphi_C\left(x_{c_i}\right)=\frac{\prod_{i=1}^{n}\phi_C\left(x_{c_i}\right)}{\prod_{j=1}^{m}\psi_S\left(x_{s_j}\right)}$$

where $x=\left(x_{c_1},\ldots,x_{c_n}\right)$ and each variable x_{c_i} correspond to a clique and x_{s_j} correspond to a separator.

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where $\boldsymbol{x} = \left(x_{c_1}, ..., x_{c_n}\right)$ and each variable x_{c_i} correspond to a clique and x_{s_j} correspond to a separator.

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Then

Main idea

- The idea is to transform one representation of the joint distribution to another in which for each clique, C, the potential function gives the marginal distribution for the variables in C, i.e.

$$\phi_C\left(x_{c_i}\right) = P\left(x_{c_i}\right)$$

- This will also apply for each separator, S.

We will have two potential functions

The ones for the Cliques

$$\phi_C\left(x_{c_i}\right)$$

The Other for the Separators

$$\psi_S\left(x_{s_i}\right)$$

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This depends on local consistency

Local Consistency

- For each two adjacent cliques U, V and their separator $S = U \cap V$:

$$\sum_{x_{U-S}} \phi_U\left(x_s, x_{U-S}\right) = \psi_S = \sum_{x_{V-S}} \phi_V\left(x_s, x_{V-S}\right)$$

And it is possible to prove that

$$p\left(x_C\right) \propto \phi_C$$
$$p\left(x_S\right) \propto \psi_S$$

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Support for this idea

Theorem

- Let probability $p(x)$ be represented by the clique potentials ϕ_C and separator potential ψ_S.

Then if the local consistence holds for each edge in the junction tree,

- Then, clique and separator are proportional to local marginal probabilities:

$$p(x_C) \propto \phi_C$$

$$p(x_S) \propto \psi_S$$

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Outline

1 Introduction
- What do we want?

2 Belief Propagation
- The Intuition
- Inference on Trees
  - The Messages
  - The Implementation

3 Junction Trees
- The Junction Tree Concept
  - Chordal Graphs
  - Maximal Clique
  - Tree Graphs
  - Junction Tree Formal Definition
  - Algorithm For Building Junction Trees
- Example
  - Moralize the DAG
  - Triangulate
  - Listing of Cliques
- Potential Function
- The Junction Tree Inference Algorithms
- Propagating Information in a Junction Tree
  - Update
  - Lemma of Propagation of Information
  - Example
- Now, the Full Propagation
  - Example of Propagation

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Now, Initialization

To initialize the potential functions (Three Steps)

1. Set all potentials to unity
2. For each variable, x_i, select one node in the junction tree (i.e. one clique) containing both that variable and its parents, $pa(x_i)$, in the original DAG.
3. Multiply the potential by $P(x_i|pa(x_i))$

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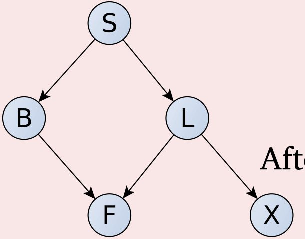
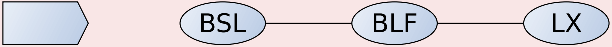
3. Multiply the potential by $P\left(x_i|pa\left(x_i\right)\right)$

Then

For example, we have at the beginning $\phi_{BSL} = \phi_{BFL} = \phi_{LX} = 1$,then using the pa



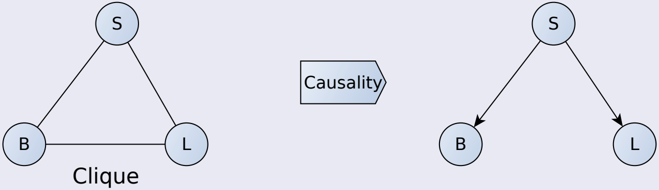


After Initialization

$$\phi_{BSL} = P(b|s)\,P(l|s)\,P(s)$$
$$\phi_{BFL} = P(f|b,l)$$
$$\phi_{LX} = P(x|l)$$

We finish with the following initial updates

$\phi_{BSL} = 1 \times P(b|s)\, P(l|s)\, P(s)$



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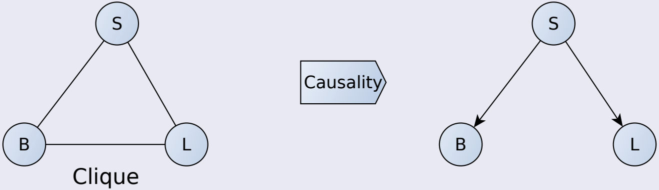

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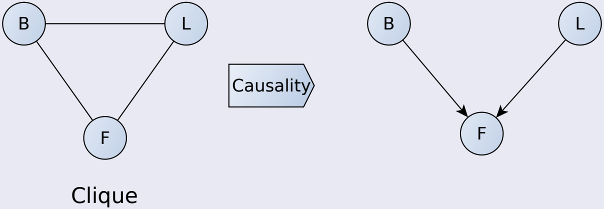
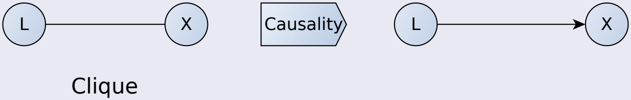


Finally

$$\phi_{LX} = 1 \times P\left(x|l\right)$$



Outline

1 Introduction
- What do we want?

2 Belief Propagation
- The Intuition
- Inference on Trees
  - The Messages
  - The Implementation

3 Junction Trees
- The Junction Tree Concept
  - Chordal Graphs
  - Maximal Clique
  - Tree Graphs
  - Junction Tree Formal Definition
  - Algorithm For Building Junction Trees
- Example
  - Moralize the DAG
  - Triangulate
  - Listing of Cliques
- Potential Function
- The Junction Tree Inference Algorithms
- Propagating Information in a Junction Tree
  - Update
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  - Example
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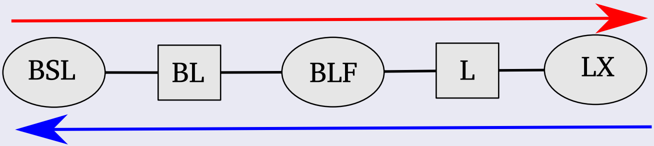
Now, we need to define the concept of propagation of information

For this, we need to pass information through the separators



Outline

1. Introduction
   - What do we want?

2. Belief Propagation
   - The Intuition
   - Inference on Trees
     - The Messages
     - The Implementation

3. Junction Trees
   - The Junction Tree Concept
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Update Information in a Junction Tree

Passing Information using the separators

- Passing information from one clique C_1 to another C_2 via the separator in between them, S, requires two steps

First Step

- Obtain a new potential for S by marginalizing out the variables in C_1 that are not in S:

$$\psi_S^* = \sum_{C_1 - S} \phi_{C_1}$$

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Propagating Information in a Junction Tree

Passing Messages in the Junction Tree

Obtain a new potential for C_2:

$$\phi^*_{C_2} = \phi_{C_2}\lambda_S$$

Where

$$\lambda_S = \frac{\psi^*_S}{\psi_S}$$

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We have the following Leamma

Lemma.The Update functions satisfies the following properties

1. The joint probability remains the same

$$\frac{\phi_{C_1}\phi_{C_2}}{\psi_S} = \frac{\phi^*_{C_1}\phi^*_{C_2}}{\psi^*_S}$$

2. $\sum_{C_1-S}\phi^*_{C_1} = \psi^*_S$
3. If $\sum_{C_2-S}\phi_{C_2} = \psi_S$ then also $\sum_{C_2-S}\phi^*_{C_2} = \psi^*_S$

Not only that

Corolary

- After UPDATE (C_1, C_2) and UPDATE (C_2, C_1) the local consistency holds for C_1 and C_2.

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- The Junction Tree Concept
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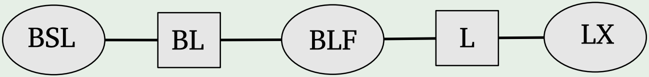
An Example

Consider a flow from the clique {B,S,L} to {B,L,F}



An Example

Initial representation

$\phi_{BSL} = P(B|S)\,P(L|S)\,P(S)$

	l_1	l_2
s_1, b_1	0.00015	0.04985
s_1, b_2	0.00045	0.14955
s_2, b_1	0.000002	0.039998
s_2, b_2	0.000038	0.759962

$\phi_{BL} = 1$

	l_1	l_2
b_1	1	1
b_2	1	1

$\phi_{BLF} = P(F|B,L)\,P(B)\,P(L) = P(F|B,L)$

	l_1	l_2
f_1, b_1	0.75	0.1
f_1, b_2	0.5	0.05
f_2, b_1	0.25	0.9
f_2, b_2	0.5	0.95

Then

After Flow

$\phi_{BSL} = P(B\|S)\,P(L\|S)\,P(S)$		
	l_1	l_2
s_1, b_1	0.00015	0.04985
s_1, b_2	0.00045	0.14955
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$\phi_{BL} = 1$		
	l_1	l_2
b_1	0.000152	0.089848
b_2	0.000488	0.909512

$\phi_{BLF} = P(F\|B, L)$		
	l_1	l_2
f_1, b_1	0.000114	0.0089848
f_1, b_2	0.000244	0.0454756
f_2, b_1	0.000038	0.0808632
f_2, b_2	0.000244	0.8640364

Now Introduce Evidence

We have

A flow from the clique $C_1 = \{B, S, L\}$ to $C_2 = \{B, L, F\}$, but this time we he information that Joe is a smoker, $E = S = s_1$.

For this, we can think on $H = V - E$

- If we assume $\overline{x}_E$ is fixed (evidence):

$$\widetilde{\phi}_{C\cap H}\left(x_{C\cap H}\right) = \phi_C\left(\underbrace{x_{C\cap H}, \overline{x}_{C\cap E}}_{x_C}\right)$$

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Slicing the Probabilities

This corresponds to taking a slice of the local function

$$\phi_{X,Y} = \begin{bmatrix} 0.12 & 0.08 \\ 0.24 & 0.56 \end{bmatrix}$$

If $E = \{Y\}$ and $\overline{y} = 1$, we get

Properties

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Then

We have that

$$
\begin{aligned}
p\left(x_H|\overline{x}_E\right) &= \frac{p\left(x_H,\overline{x}_E\right)}{p\left(\overline{x}_E\right)}\\
&= \frac{\frac{1}{z}\prod_C \phi_C\left(x_{C\cap H},\overline{x}_{C\cap E}\right)}{\sum_H \frac{1}{z}\prod_C \phi_C\left(x_{C\cap H},\overline{x}_{C\cap E}\right)}\\
&= \frac{\prod_C \tilde{\phi}_{C\cap H}\left(x_{C\cap H}\right)}{\sum_H \prod_C \tilde{\phi}_{C\cap H}\left(x_{C\cap H}\right)}\\
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Example

Incorporation of Evidence

$\phi_{BSL} = P(B|S)\,P(L|S)\,P(S)$

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An Example

After Flow

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b_2	0.00045	0.14955

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	l_1	l_2
f_1, b_1	0.0001125	0.004985
f_1, b_2	0.000245	0.0074775
f_2, b_1	0.0000375	0.044865
f_2, b_2	0.000255	0.1420725

Outline

1. Introduction
   - What do we want?
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   - The Intuition
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     - The Messages
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3. Junction Trees
   - The Junction Tree Concept
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The Full Propagation

Two phase propagation (Jensen et al, 1990)

1. Select an arbitrary clique, C_0
2. Collection Phase – flows passed from periphery to C_0
3. Distribution Phase – flows passed from C_0 to periphery

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3 Junction Trees
- The Junction Tree Concept
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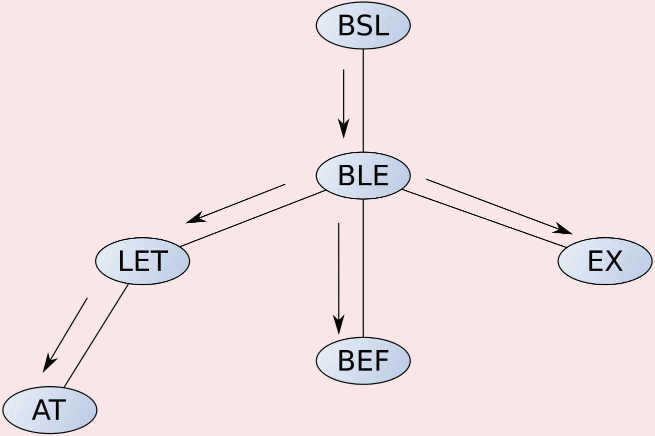
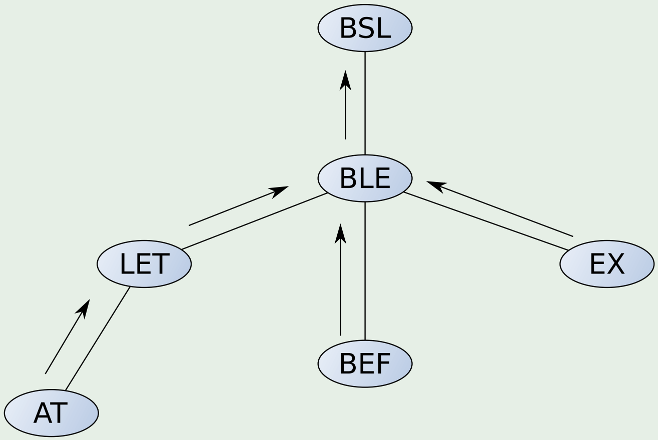
Example

Distribution



Example

Collection



The Full Propagation

After the two propagation phases have been carried out

- The Junction tree will be in equilibrium with each clique containing the joint probability distribution for the variables it contains.
- Marginal probabilities for individual variables can then be obtained from the cliques.

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After propagation the result will be

$$P\left(x, E\right)=\frac{\prod_{c\in C}\phi_{c}\left(x_{c}, E\right)}{\prod_{s\in S}\psi_{s}\left(x_{s}, E\right)}$$

After normalization

$$P\left(x|E\right)=\frac{\prod_{c\in C}\phi_{c}\left(x_{c}|E\right)}{\prod_{s\in S}\psi_{s}\left(x_{s}|E\right)}$$

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