Artificial Intelligence

Causality in Bayesian Networks

Andres Mendez-Vazquez

February 11, 2020

Outline

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Causality

What do we naturally?

A way of structuring a situation for reasoning under uncertainty is to construct a graph representing causal relations between events.

Example of events with possible outputs

- Fuel? {Yes, No}
- Clean Spark Plugs? {full, 1/2, empty}
- Start? {Yes, No}

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- Fuel? {Yes, No}
- Clean Spark Plugs? {full, $1/2$, empty}
- Start? {Yes, No}

Causality

We know

We know that the state of **Fuel?** and the state of **Clean Spark Plugs?** have a causal impact on the state of **Start?**.

Thus we have something like

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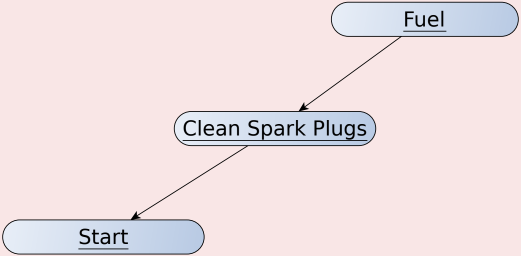

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Causal Structure - Judea Perl (1988)

Definition

A causal structure of a set of variables V is a directed acyclic graph (DAG) in which each node corresponds to a distinct element of V, and each edge represents direct functional relationship among the corresponding variables.

Observation

Causal Structure $\approxeq$ A precise specification of how each variable is influenced by its parents in the DAG.

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A causal model is a pair $M = \langle D, \Theta_D \rangle$ consisting of a causal structure D and a set of parameters Θ_D compatible with D.

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The parameters Θ_D assign a distribution

$$x_i = f_i\left(\mathsf{pa}_i, u_i\right)$$
$$u_i \sim p\left(u_i\right)$$

- pa_i are the parents of x_i in D.
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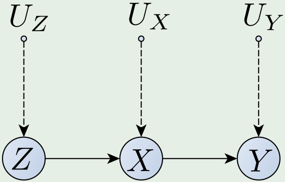
Example

From the point of view of Statistics



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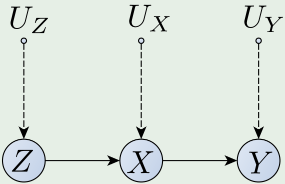

Formulation

$$z = f_Z\left(u_Z\right)$$

$$x = f_X\left(z, u_X\right)$$

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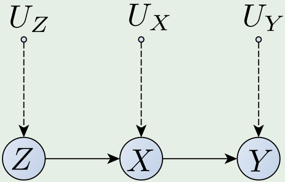

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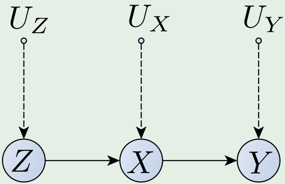

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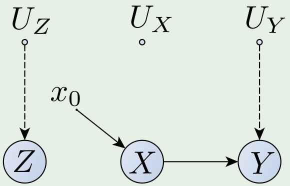

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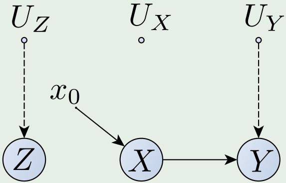

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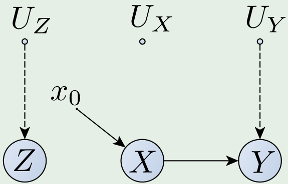

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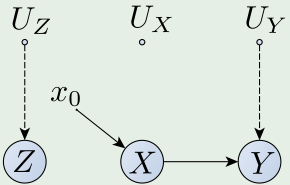

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In order to analyze a causal network is necessary to analyze:

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Causal Chains

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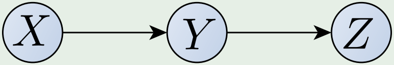

X : Low Pressure

Y : Rain

Z : Traffic

What about the Joint Distribution?

We have by the Chain Rule

$$P(X, Y, Z) = P(X)\, P(Y|X)\, P(Z|Y, X) \quad (1)$$

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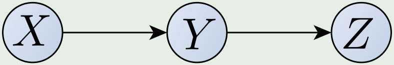

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Propagation of Information

Given no information about Y

Information can propagate from X to Z.

Thus

The natural question is What does happen if $Y = y$ for some value y?

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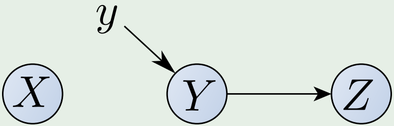
Blocking Propagation of Information



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Using Our Probabilities

Then Z is independent of X given a $Y = y$

And making the assumption that once an event happens

$$P(Z|X, Y = y) = P(Z|Y = y) \quad (2)$$

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$$P(Z|X, Y = y) = P(Z|Y = y) \quad (2)$$

YES!!!

Evidence along the chain "blocks" the influence.

Thus

Something Notable

Knowing that X has occurred does not make any difference to our beliefs about Z if we already know that Y has occurred.

Thus conditional independencies can be written

$$I_P(Z, X|Y = y) \quad (3)$$

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Therefore

The Joint Probability is equal to

$$P(X, Y = y, Z) = P(X)\, P(Y = y|X)\, P(Z|Y = y) \tag{4}$$

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Thus

Is X independent of Z given $Y = y$?

$$P(Z|X, Y = y) = \frac{P(X, Y = y, Z)}{P(X, Y = y)}$$

$$= \frac{P(X)\, P(Y = y|X)\, P(Z|Y = y)}{P(X)\, P(Y = y|X)}$$

$$= P(Z|Y = y)$$

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Thus

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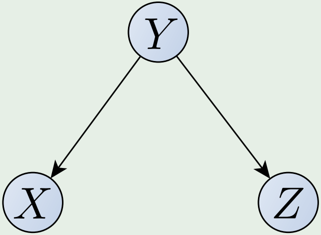
Common Causes

Another basic configuration: two effects of the same cause

X : John Calls
Y : Alarm
Z : Mary Calls



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$$P(X, Y = y, Z) = P(X)\, P(Y = y|X)\, \underbrace{P(Z|X, Y = y)}_{P(Z|Y=y)} \quad (5)$$

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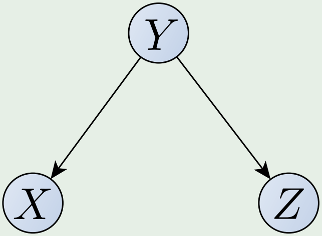

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Common Causes

What happened if X is independent of Z given $Y = y$?

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Evidence on the top of the chain "blocks" the influence between X and Z

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It gives rise to the same conditional independent structure as chains

$$I_P\left(Z, X | Y = y\right) \tag{6}$$

i.e.

if we already know about Y, then an additional information about X will not tell us anything new about Z.

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Common Effect

Last configuration: two causes of one effect (v-structures)

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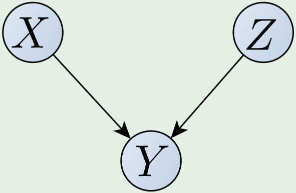
Y : Traffic
Z : Ballgame



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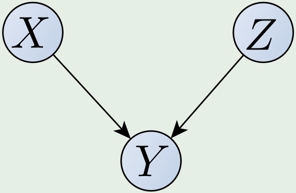

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Proof

We have the following

$$P(Z|X,Y) = \frac{P(X,Y,Z)}{P(X,Y)}$$

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Common Effects

Are X and Z independent given $Y = y$?

No!!! Because seeing traffic puts the rain and the ballgame in competition as explanation!!!

$$P\left(X, Z | Y = y\right)$$

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The General Case

Backwards from the other cases

- Observing an effect activates influence between possible causes.

Fact

Any complex example can be analyzed using these three canonical cases

Question

In a given Bayesian Network, Are two variables independent (given evidence)?

Solution

- **Analyze Graph Deeply!!!**

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Analyze the Graph

Definition 2.1

- Let $G = (V, E)$ be a DAG, where V is a set of random variables. We say that, based on the Markov condition, G entails conditional independence:
  - $I_P(A, B|C)$ for $A, B, C \subseteq V$ if $I_P(A, B|C)$ holds for every $P \in P_G$,

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Examples of Entailed Conditional independence

Example

G: Graduate Program Quality.
F: First Job Quality.
B: Number of Publications.
C: Number of Citations.

F is given some evidence!!!

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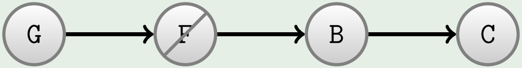

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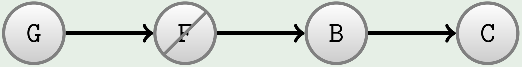

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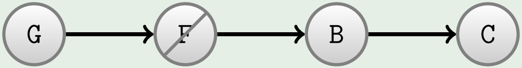

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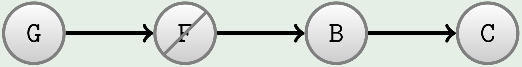
Using Markov Condition

If the graph satisfies the Markov Condition



Thus

$$\begin{aligned} P(C|G, F = f) &= \sum_{b} P(C|B = b, G, F = f)\, P(B = b|G, F = f) \\ &= \sum_{b} P(C|B = b, F = f)\, P(B = b|F = f) \\ &= P(C|F = f) \end{aligned}$$

Finally

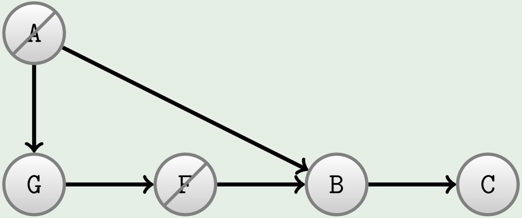
We have

$$I_p\left(C, G|F\right) \tag{7}$$

D-Separation ≈ Conditional independence

F and G are given as evidence

Example C and G are d-separated by A, F in the DAG in



Outline

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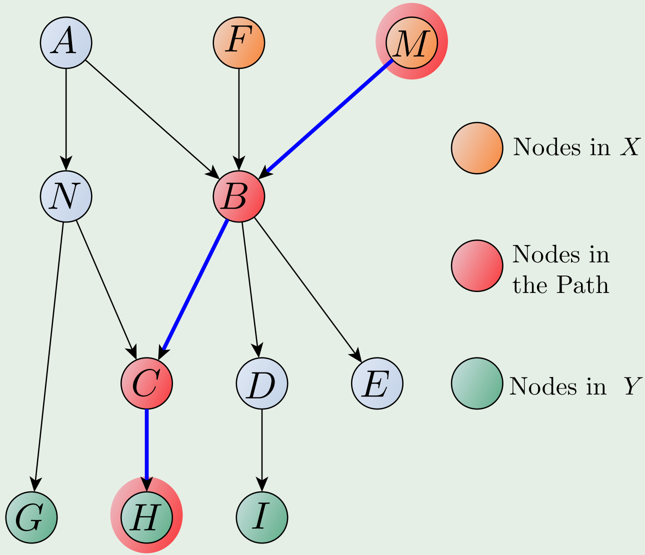
Basic Definitions

Definition (Undirected Paths)

A path between two sets of nodes X and Y is any sequence of nodes between a member of X and a member of Y such that every adjacent pair of nodes is connected by an edge (regardless of direction) and no node appears in the sequence twice.

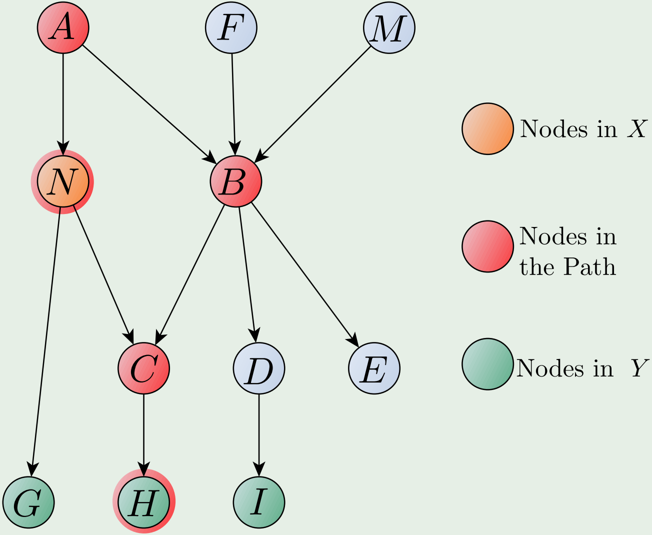
Example

An example



Example

Another one



Thus

Given a path in $G = (V, E)$

There are the edges connecting $[X_1, X_2, ..., X_k]$.

Therefore

Given the directed edge $X \to Y$, we say the tail of the edge is at X and the head of the edge is Y.

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Given the directed edge $X \rightarrow Y$, we say the tail of the edge is at X and the head of the edge is Y.

Basic Classifications of Meetings

Head-to-Tail

A path $X \to Y \to Z$ is a **head-to-tail meeting**, the edges meet head-to-tail at Y, and Y is a head-to-tail node on the path.

Tail-to-Tail

A path $X \leftarrow Y \to Z$ is a **tail-to-tail meeting**, the edges meet tail-to-tail at Z, and Z is a tail-to-tail node on the path.

Head-to-Head

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Examples

Head-to-Tail

X : Low Pressure

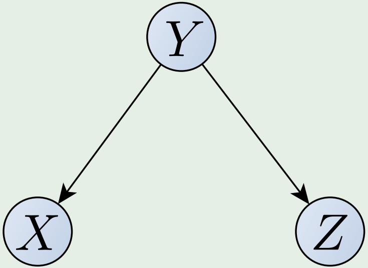
Y : Rain

Z : Traffic

Examples

Tail-to-Tail

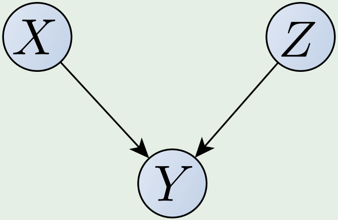
X : John Calls
Y : Alarm
Z : Mary Calls



Examples

Head-to-Head

X : Raining
Y : Traffic
Z : Ballgame



Basic Classifications of Meetings

Finally

- A path (undirected) $X - Z - Y$, such that X and Y are not adjacent, is an uncoupled meeting.

Outline

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Blocking Information ≈ Conditional Independence

Definition 2.2

Definition 2.2 Let$G = (V, E)$ be a DAG, $A \subseteq V$, X and Y be distinct nodes in $V - A$, and ρ be a path between X and Y .

Then ρ is **blocked** by A if one of the following holds:

1. There is a node $Z \in A$ on the path ρ, and the edges incident to Z on ρ meet head-to-tail at Z.
2. There is a node $Z \in A$ on the path ρ, and the edges incident to Z on ρ, meet tail-to-tail at Z.
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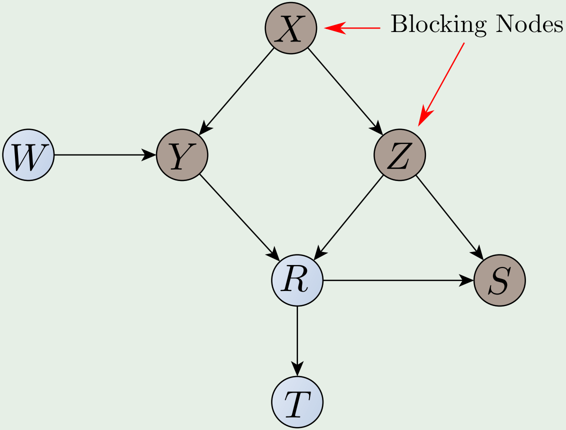
Example

We have that the path $[Y, X, Z, S]$ is blocked by $\{X\}$ and $\{Z\}$

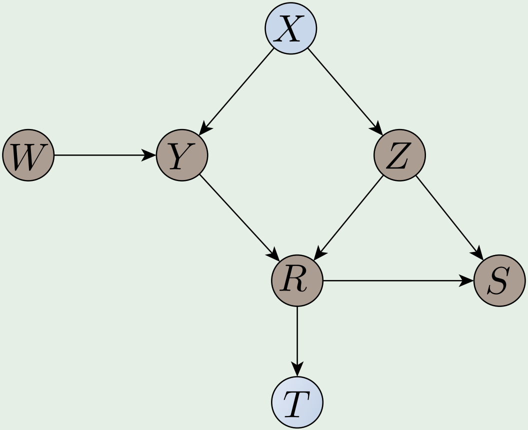
- Because the edges on the chain incident to X meet tail-to-tail at X.



Example

We have that the path $[W, Y, R, Z, S]$ is blocked by $\emptyset$

- Because $R \notin \emptyset$ and $T \notin \emptyset$ and the edges on the chain incident to R meet head-to-head at R .



Outline

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Definition of D-Separation

Definition 2.3

Let $G = (V, E)$ be a DAG, $A \subseteq V$, and X and Y be distinct nodes in $V - A$. We say X and Y are **D-Separated** by A in G if every path between X and Y is **blocked** by A.

Definition 2.4

Let $G = (V, E)$ be a DAG, and A, B, and C be mutually disjoint subsets of V. We say A and B are d-separated by C in G if for every $X \in A$ and $Y \in B$, X and Y are **D-Separated** by C.

We write

$$I_G(A, B|C) \text{ or } A \perp B|C \tag{8}$$

If $C = \emptyset$, we write only $I_G(A, B)$ or $A \perp B$.

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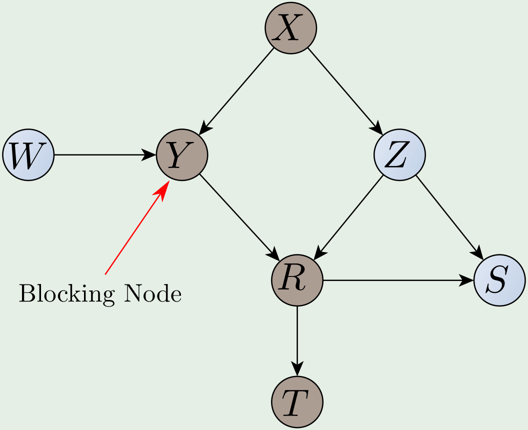
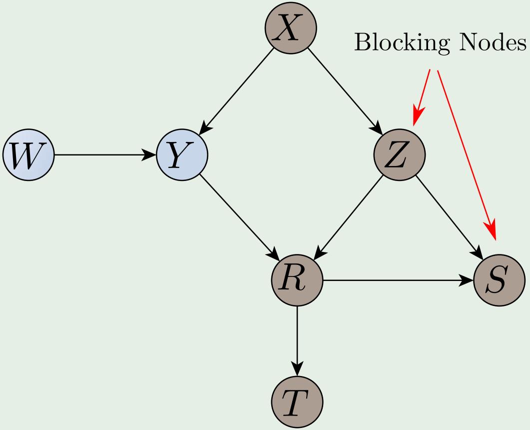


Example

X and T are D-Separated by $\{Y, Z\}$ - It is the set that block all paths

- Because the chain $[X, Z, S, R, T]$ is blocked at Z, S.



D-Separation ⇒ Independence

D-Separation Theorem

Let P be a probability distribution of the variables in V and $G = (V, E)$ be a DAG. Then (G, P) satisfies the Markov condition if and only if

- for every three mutually disjoint subsets $A, B, C \subseteq V$, whenever A and B are D-Separated by C, A and B are conditionally independent in P given C.

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Proof

The proof that, if (G, P) satisfies the Markov condition

Then, each D-Separation implies the corresponding conditional independence is quite lengthy and can be found in [Verma and Pearl, 1990] and in [Neapolitan, 1990].

Then, we will only prove the other direction

Suppose each D-Separation implies a conditional independence.

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Something Notable

It is not hard to see that a node's parents D-Separate the node from all its non-descendent's that are not its parents.

This is

If we denote the sets of parents and non-descendent's of X by PA_X and ND_X respectively, we have

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Every Entailed Conditional Independence is Identified by D-Separation

Lemma 2.2

Any conditional independence entailed by a DAG, based on the Markov condition, is equivalent to a conditional independence among disjoint sets of random variables.

Theorem 2.1

Based on the Markov condition, a DAG G entails all and only those conditional independences that are identified by D-Separations in G.

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Outline

Now

We would like to find the D-Separations

Since d-separations entail conditional independencies.

We want an efficient algorithm

For determining whether two sets are D-Separated by another set.

For This, we need to build an algorithm

One that can find all D-Separated nodes from one set of nodes by another.

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To accomplish this

We will first find every node X such that there is at least one active path given A between X and a node in D.

Something like

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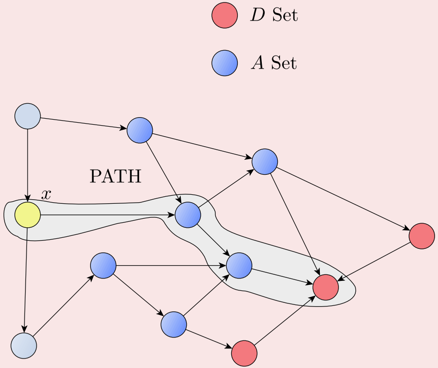

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We can find the set R of all nodes reachable from x as follows

Any node V such that the edge $x \to v$ exists is reachable.

Then

We label such edge with 1.

Next for each such v

We check all unlabeled edges $v \to w$ and see if $(x \to v, v \to w)$ is a legal pair.

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And keep going!!!

Similar to a Breadth-First Graph

Here, we are visiting links rather than nodes.

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Then

If there is an active path ρ between node X and some other node

Then every 3-node sub-path $u - v - w$ $(u \neq w)$ of ρ has the following property.

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The algorithm find-reachable-nodes uses the RULE

Find if $(u \to v, v \to w)$ is legal in G'.

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Outline

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$in_A[v]$ is false and $descendent[v]$ is false for all $v \in V$

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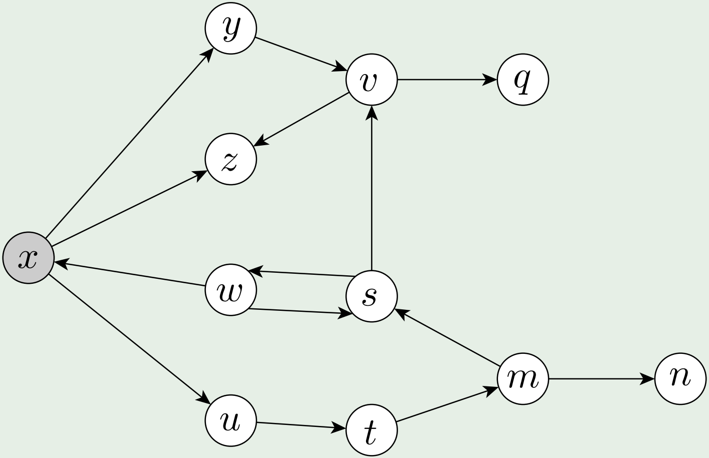
Example

Labeling edges to 1 and shaded nodes are in R



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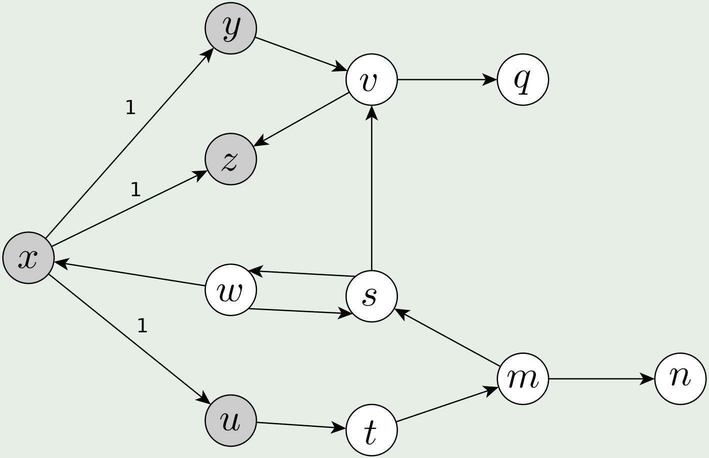
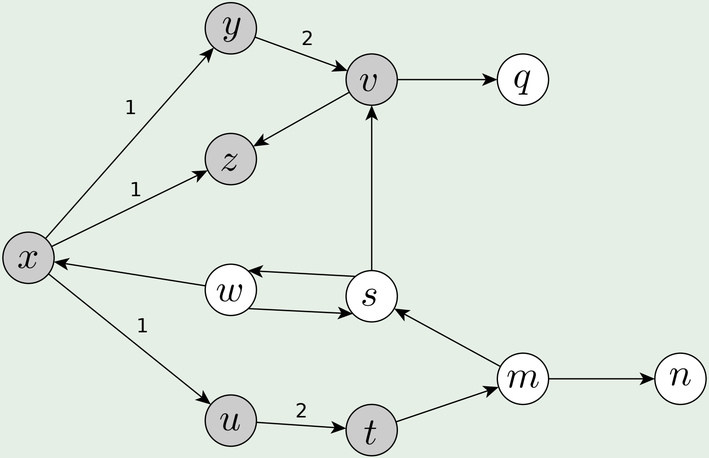


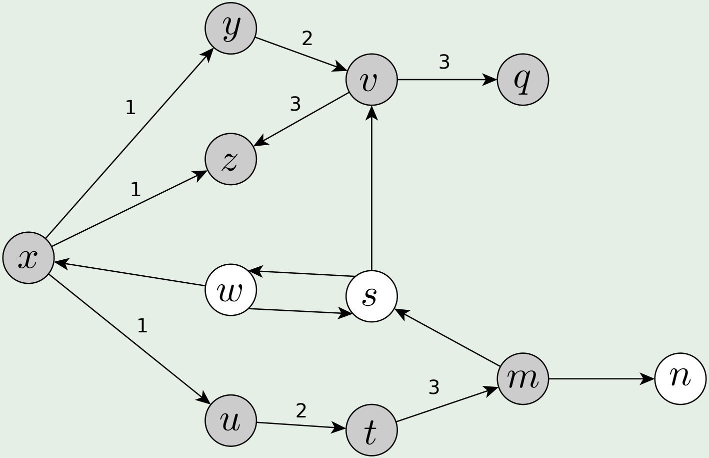
Example

Labeling edges to 2 and shaded nodes are in R



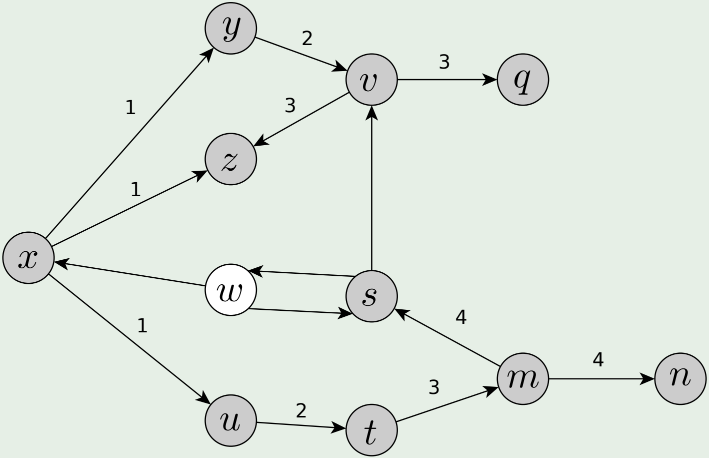
Example

Labeling edges to 3 and shaded nodes are in R



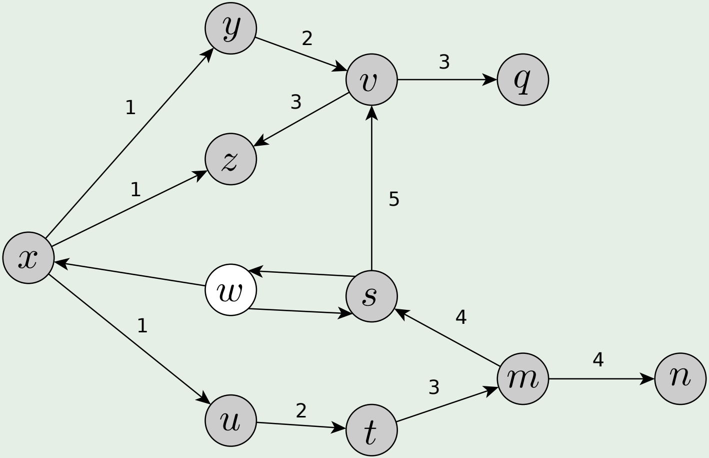
Example

Labeling edges to 4 and shaded nodes are in R



Example

Labeling edges to 5 and shaded nodes are in R



Outline

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Algorithm to Finding Reachability

find-reachable-nodes(G, set of nodes B, set of nodes $\&R$)

Input: G = (V,E), subset B $\subset$ V and a RULE to find if two consecutive edges are legal
Output: R $\subset$ V of all nodes reachable from B

1. for each $x \in$ B
2. add x to R
3. for (each v such that $x \rightarrow v \in$ E)
5. add v to R
6. label $x \rightarrow v$ with 1
7. $i = 1$
8. $found$ =true

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        for (each v such that x → v ∈ E)
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```

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8. $found$ =true

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Continuation

The following steps

9. while $(found)$
10. $\quad found =$false
11. $\quad$ for (each v such that $u \rightarrow v$ labeled i)
12. $\quad\quad$ for (each *unlabeled* edge $v \rightarrow w$
13. $\quad\quad\quad$ such $(u \rightarrow v, v \rightarrow w)$ is legal)
14. $\quad\quad\quad\quad$ add w to R
15. $\quad\quad\quad\quad$ label $v \rightarrow w$ with $i + 1$
16. $\quad\quad\quad\quad$ $found =$true
17. $\quad$ $i = i + 1$

Continuation

The following steps

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16. $found$ =true
17. $i = i + 1$

Continuation

The following steps

```
while (found)
    found = false
    for (each v such that u → v labeled i)
        for (each unlabeled edge v → w
            such (u → v, v → w) is legal)
                add w to R
                label v → w with i + 1
                found = true
    i = i + 1
```

Outline

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Complexity of find-reachable-nodes

We have that

Let n be the number of nodes and m be the number of edges.

Something Notable

In the worst case, each of the nodes can be reached from n entry points.

Thus

Each time a node is reached, an edge emanating from it may need to be re-examined.

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$$W(m,n) = \Theta(mn) \tag{15}$$

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Outline

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Algorithm for D-separation

Find-D-Separations($DAG\ G =$ (V,E) , set of nodes A,B, set of nodes D)

Input: $G =$ (V,E)and two disjoint subsets A,B $\subset$ V

Output: D $\subset$ V containing all nodes D-Separated from every node in B by A. That is I_G (B,D|A) holds and no superset D has this property.

1. for each $v \in$ V
2. if ($v \in$ A)
3. $in_A[v] = true$
4. else
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6. if (v is or has a descendent in A)
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10. E$'$ = E $\cup$ $\{u \to v | v \to u \in$ E$\}$
11. G' = (V,E$'$)
12. Run the algorithm:

    find-reachable-nodes(G', B,R)

    $\triangleright$ Note B $\subseteq$ R
13. return D=V $-$ (A $\cup$ R)

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10. E′ = E ∪ $\{u \to v | v \to u \in$ E$\}$
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12. Run the algorithm:

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Observation about $descendent\,[v]$

We can implement the construction of $descendent\,[v]$ as follow

Initially set $descendent\,[v] =$ true for all nodes in A.

Then

Then follow the incoming edges in A to their parents, their parents' parents, and so on.

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We set $descendent\,[v] =$ true for each node found along the way.

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The RULE about legal and E'

The E' is necessary because using only the RULE on E will no get us all the active paths.

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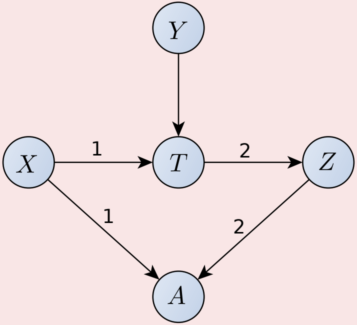

Thus

Something Notable

Given A is the only node in A and $X \to T$ is the only edge in B, the edges in that DAG are numbered according to the method.

Then

The active chain $X \to A \leftarrow Z \leftarrow T \leftarrow Y$ is missed because the edge $T \to Z$ is already numbered by the time the chain $A \leftarrow Z \leftarrow T$ is investigated.

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If we use the set of edges E'.

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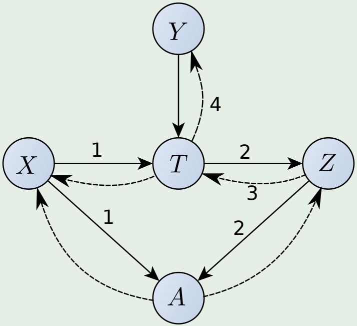
Once, we add the extra edges, we get



Outline

Complexity

Please take a look at page 81 at

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For the analysis of the algorithm for m edges and n nodes

$$\Theta(m) \text{ with } m \geq n. \tag{16}$$

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The D-Separation Algorithm works

Theorem 2.2

The set D contains all and only nodes D-Separated from every node in B by A.

- That is, we have I_G (B,D|A) and no superset of D has this property.

Proof

- The set R determined by the algorithm contains
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Proof

For any two nodes $x \in$ B and $y \notin$ A $\cup$ B

The path $x - ... - y$ is active in G if and only if the path $x \rightarrow ... \rightarrow y$ is legal in G'.

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Thus R contains the nodes in B plus all and only those nodes that have active paths between them and a node in B.

By the definition of D-Separation

A node is D-Separated from every node in B in A if the node is not in A $\cup$ B and there is not active path between the node and a node in B.

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Thus

D=V-$(A \cup R)$ is the set of all nodes D-Separated from every node in B by A.

Outline

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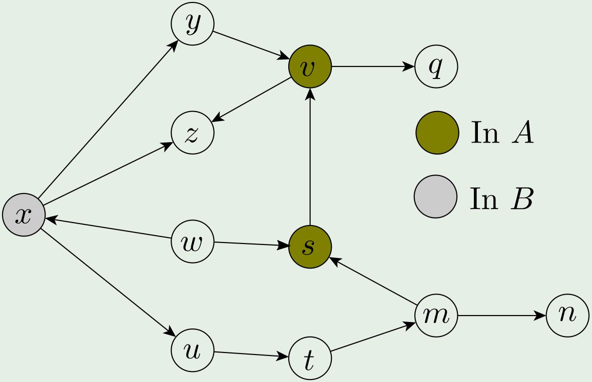
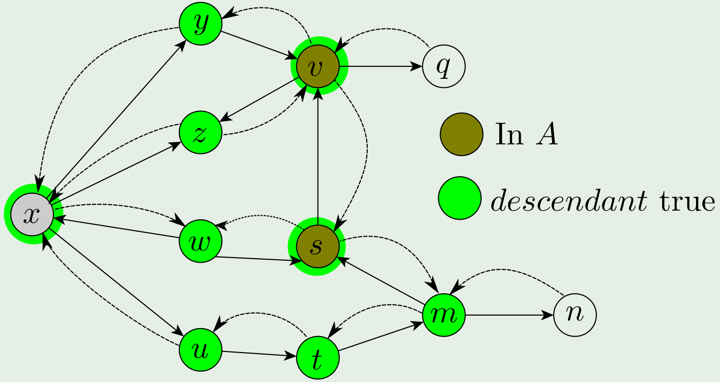
Example

$B = \{x\}$ and $A = \{s, v\}$- Original Graph!!!



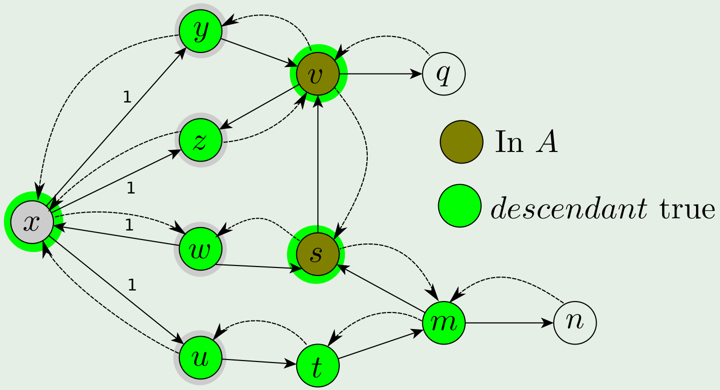
Example

$B = \{x\}$ and $A = \{s, v\}$- Moralized and with the tracking of descendants



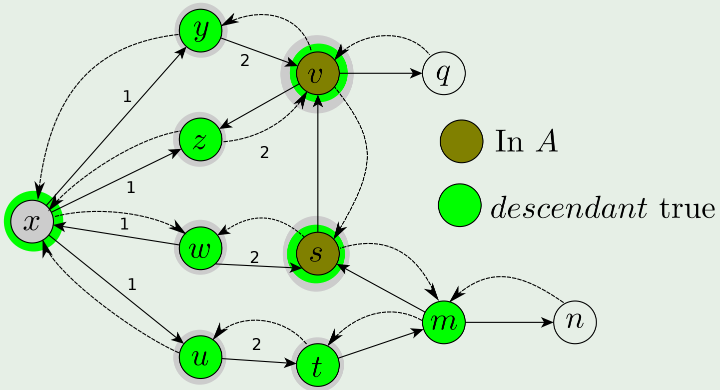
Example

$B = \{x\}$ and $A = \{s, v\}$ and the first part of the reachability algorithm



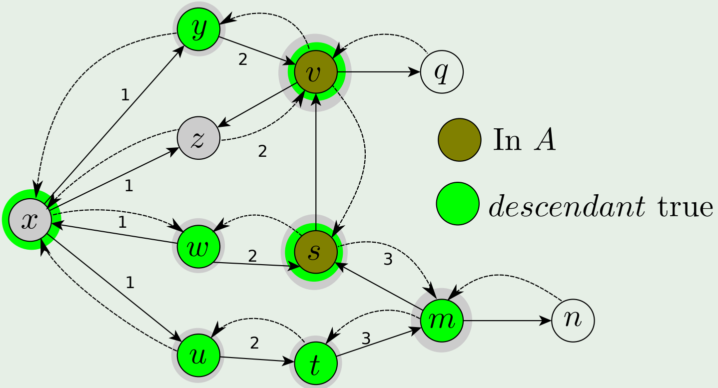
Example

Remember that Legality is in G



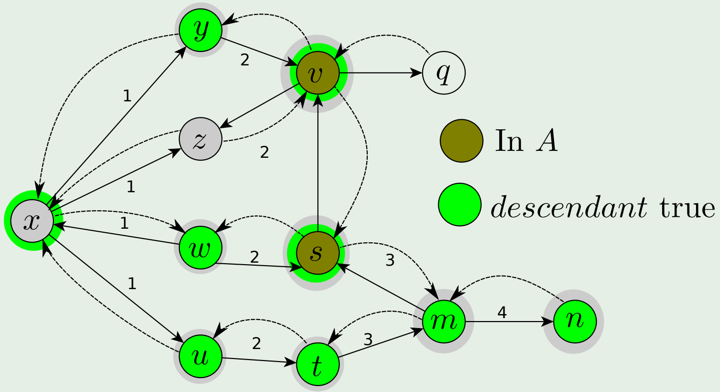
Example

Now, we have that



Example

$D - \{A \cup R\} = \{q\}$



Outline

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Application

Something Notable

In general, the inference problem in Bayesian networks is to determine $P(B|A)$, where A and B are two sets of variables.

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We can use the D-Separation for that.

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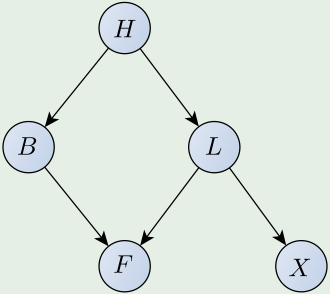
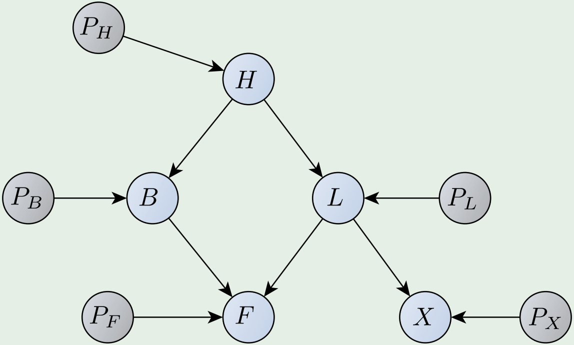
Given the following the DAG G



Example

We generate G'



Where

The extra nodes represent

The probabilities in the interval $[0,1]$ and representing $P(X=x)$

Creating a set of P be the set of auxiliary parent nodes

Thus, if we want to determine $P(\text{B}|\text{A})$ in G, we can use the algorithm for D-Separation to find D.

Such that

$$I_{G'}(\text{B,D}|\text{A}) \quad (17)$$

And no superset of D has this property, then take D∩P.

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For example

We want $P(f)$

To determine it we need that all and only the values of P_H , P_B , P_L , and P_F

Because

$$I_{G'}(\{F\}, \{P_X\} | \emptyset) \quad (18)$$

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We want $P(f|b)$

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Then

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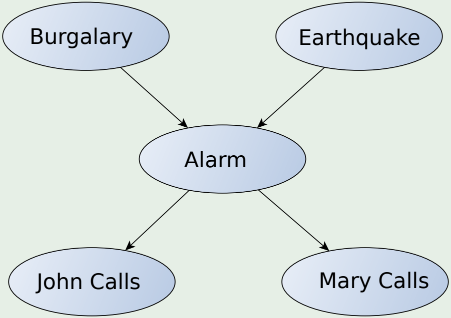

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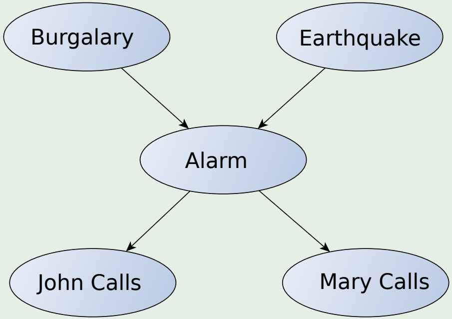

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