Introduction to Artificial Intelligence

Backtracking

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April 23, 2019

Outline

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In the Beginning

James Bernoulli 17th Century [1]

- He successfully used the principle to solve the “Tot tibi sunt dotes”

Basically a Combinatoric Problem

- How many ways exist to write such words by permuting them?

tibi sunt tot dotes

sunt tibi tot dotes

...

His notes with the traces exist

- They look as a classic backtracking algorithm...

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Around 1882

- Edourdad Lucas credited his student Tremaux
  - About the use of depth-first search in walk of a tree...

The problem of the 8 queens was first proposed in 1948-1950

- By Max Bezzel and Franz Nauck

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And in the letter dated 27 of September 1850

Gauss explained how to find all the solutions by Backtracking

- He called the procedure “Tatonniren” from French “to feel one’s way.”

Finally

Computer arrived finally 100 years latter

- And the technique was fully described by Robert J. Walker

He introduced a full description of Backtracking

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n Chess Queen Problem

A Puzzle

- Given a $n \times n$ chessboard, How to position n queens such that they cannot attack each other?

Remember

- The Queen attacks any piece in the same row, column or either diagonal.

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The First Step

Starting from an empty board

- Place a Queen in the first column... then the second row... and so on

Then after doing so

- We try in the next column and proceed recursively

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The Backtracking Step

If we get stuck at some column k

- Then, we **backtrack** to the previous column $k - 1$

Observe the tree is constructed in Preorder

- We do calculations before we move to any possible options...

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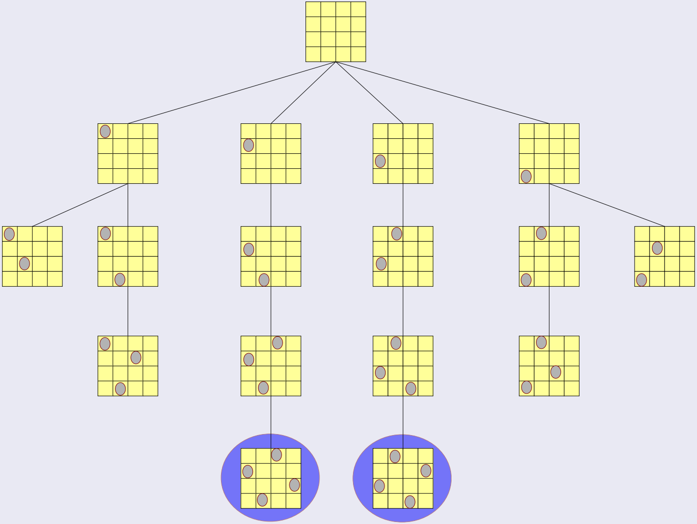
Example

Something Notable



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Codification

How do we get compact and easy to use?

Position Column	1	2	3	4
Row Position	2	4	1	3

A simple vector

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What about?

- We need to have a way to know when a queen can attack another.

We can have three boolean arrays

Array A

- It indicates if a row does not contain a queen.

Array B

- It indicates if a front diagonal does not contain a queen
  - Indexation is the sum of row and columns.

Array C

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With the following indexes

For the array A

- The indexing goes from 1 to 4.

For the array B

- The indexing goes from 2 to 8.

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- The indexing goes from -3 to 3.

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We have something quite interesting

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- The sum of the row and column indices is constant along diagonals
- The difference of the row and column indices is constant along diagonals

We can see that in the 4 Queen chessboard

- Take a look at the board...

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We can see that in the 4 Queen chessboard

- Take a lock at the board...

Code for the 8 Queens

Queen$(col : \mathbb{N})$:

local $row : \mathbb{N}$

Init a, b, c to true

for $row = 1$ **to** 8

    if $a[row]$ **and** $b[row + col]$ **and** $c[row - col]$ **then**

        $x[col] = row$

        $a[row] = b[row + col] = c[row - col] = False$

        if $col < 8$ **then Queen**$(col + 1)$ **else Print** x

        $a[row] = b[row + col] = c[row - col] = True$

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Some Notes

Properties

- It uses Depth-First Search.
- It takes a sequence $V = \{x_1, x_2, ..., x_n\}$ of variables of X to be instantiated (Initially X including all the variables).
- An initially empty instantiation I as arguments.

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Here, we can use something quite elegant

The use of recursion

- Here, we can use a recursion algorithm that at a certain node with assignaimets

$$p = \langle x_1 = a_1, x_2 = a_2, ..., x_j = a_j \rangle$$

Then, a new variable is added and a search in the possible values done

$$p' = \langle x_1 = a_1, x_2 = a_2, ..., x_j = a_j, x_{j+1} \rangle$$

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Recursive Backtracking Algorithm

BackTracking(V, I)

```
If V = ∅ then
    I is a solution
else
    Let x_i ∈ V
    for each v ∈ D_{x_i} do
        If I ∪ {(x_i, v)} is consistent then
            BackTracking (V − {x_i}, I ∪ (x_i, v))
```

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Extra memory

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Here, we have the following set

- $S_k = \langle s = x_1, x_2, ..., x_k \rangle$ a sequence of states

A always a Boolean statement

$$P\left(x_1, x_2, ..., x_k\right)$$

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Iterative Backtracking Algorithm

BackTracking

$k = 1$

Generate S_1 a stack with the initial states

while $k > 0$:

  while $S_k \neq \emptyset$

    $\triangleright$ Advance to next position

    $x_k = pop(S)$

    $p_k = \langle x_1, x_2, ..., x_{k-1} \rangle \circ x_k$

    If $P(p_k)$ then return p_k

    $T =$ Generate the new expansion from x_k

    $k = k + 1$

    Generate new stack S_k

    $push(T, S_k)$

  $k = k - 1$

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Backtracking

Something Notable

Backtracking is based on that it is often possible to reject a solution by looking at just a small portion of it.

Example

If an instance of SAT contains the clause $C_i = (x_1 \vee x_2)$, then all assignments with $x_1 = x_2 = 0$ can be instantly eliminated.

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Example

Pruning Example

Given the possible values that you can give to two literals:

x_1	x_2
1	1
1	0
0	1
0	**0**

It is possible to prune a quarter of the entire search space... Can this be systematically exploited?

An example of exploiting this idea in SAT solvers

Consider the following Boolean formula $\phi(w,x,y,z)$

$(w \lor x \lor y \lor z) \land (w \lor \neg x) \land (x \lor \neg y) \land (y \lor \neg z) \land (z \lor \neg w) \land (\neg w \lor \neg z)$

We start branching in one variable, we can choose w

Note: This selection does not violate any of the clauses of $\phi(w,x,y,z)$

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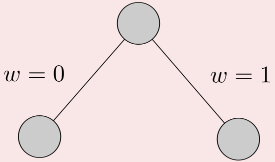

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The partial assignment $w = 0, x = 1$ violates the clause $(w \vee \neg x)$

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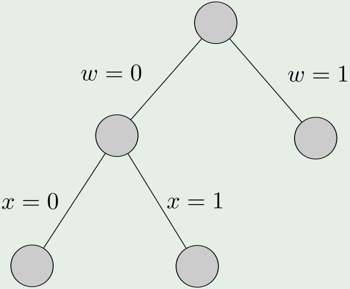

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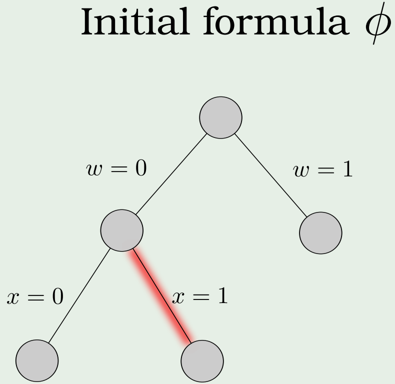
Then, we prune that branch



In addition

What if $w = 0, x = 0$

Then, the following clauses are satisfied

1. $\neg w = 1$
2. $\neg x = 1$

Thus, we have the following left

1. Before
   1. $(w \vee x \vee y \vee z) \wedge (w \vee \neg x) \wedge (x \vee \neg y) \wedge (y \vee \neg z) \wedge (z \vee \neg w) \wedge (\neg w \vee \neg z)$
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Finally

We have the following reduced number of equations

$$(y \vee z), (1), (\neg y), (y \vee \neg z), (1), (1) \Leftrightarrow \boldsymbol{(y \vee z), (\neg y), (y \vee \neg z)}$$

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Which subproblem to expand next.

Second

Which branching variable to use.

Remark

The benefit of backtracking lies in its ability to eliminate portions of the search space.

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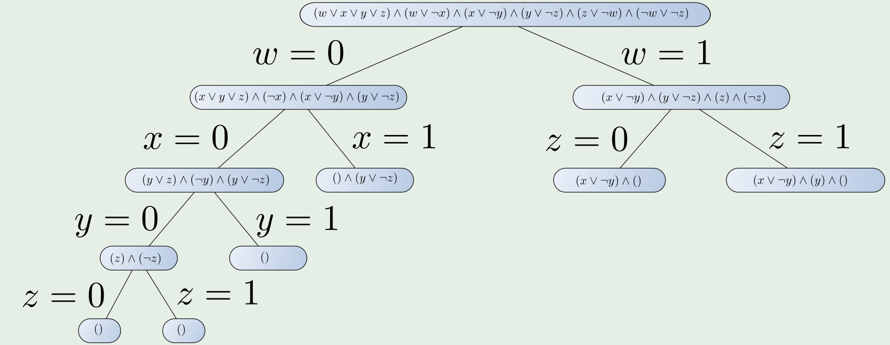

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Pseudo-code for Backtracking

We have

BACKTRACKING(P_0)

1. Start with some problem P_0
2. Let $\mathcal{S} = \{P_0\}$, the set if active subproblems
3. While $\mathcal{S} \neq \emptyset$
4. **choose** a subproblem $P \in \mathcal{S}$ and remove it from $\mathcal{S}$
5. **expand** it into smaller subproblems $P_1, P_2, ..., P_k$
6. For each P_i
7. if test(P_i) succeeds: halt and return the branch solution
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With the right test, expand, and choose routines

- Backtracking can be remarkably effective in practice

Further

- The backtracking algorithm we showed for SAT is the basis of many successful satisfiability programs

For Example, 2-SAT problems

- It is a conjunction (a Boolean and operation) of clauses,
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D. E. Knuth, *The Art of Computer Programming, Volume 4A: Combinatorial Algorithms, Part 2.*
Pearson Education India, 2011.