Introduction to Artificial Intelligence

Linear Algebra

Andres Mendez-Vazquez

January 16, 2020

Outline

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Introduction

What is this class about?

- It is clear that the use of mathematics is essential for Artificial Intelligence.

Therefore...

- The understanding of Mathematical Modeling is part of the deal...

If you want to be

- A Good Practitioner of Artificial Intelligence!!!

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Example

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A web surfer moves from a web page to another web page...

- Question: How do you model this?

You can use a graph!!!

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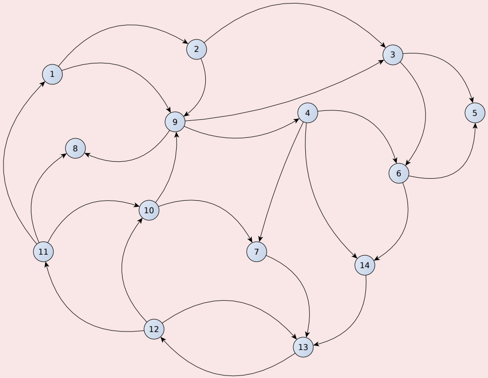
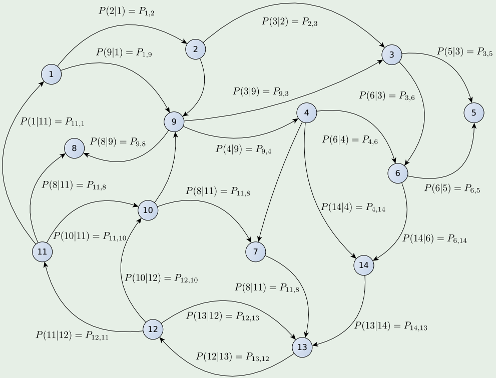


Now

Add Some Probabilities



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Thus

We can build a matrix

$$M = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{pmatrix} \quad (1)$$

Thus, it is possible to obtain certain information by looking at the eigenvector and eigenvalues

These vectors $v_\lambda' s$ and values $\lambda' s$ have the property that

$$Mv_\lambda = \lambda v_\lambda \quad (2)$$

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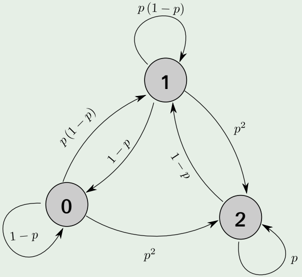
This is the Basis of Page Rank in Google

Imagine a small Web!!! Really Small...



We have the following Stochastic Matrix

$$P = \begin{pmatrix} 1-p & 1-p & 0 \\ p(1-p) & p(1-p) & 1-p \\ p^2 & p^2 & p \end{pmatrix} \quad (3)$$

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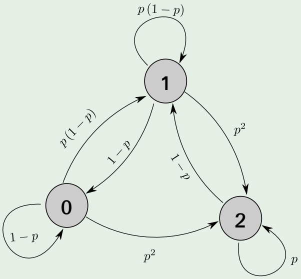

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Therefore, we can use that

To describe the transition in the Markov Chain

Let $\boldsymbol{p}_t \in \mathbb{R}^n$ is the distribution matrix of X_t at time t

$$\left(\boldsymbol{p}_t\right)_i = P\left(X_t = i\right) \quad (4)$$

Then moving from a distribution to another one we have

$$\boldsymbol{p}_{t+1} = \boldsymbol{P}\boldsymbol{p}_t \quad (5)$$

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Here, the Idea of Positive Matrix

Basic Definition

- A matrix is called positive if all its entries are positive.
- Non-negative, if all its entries are non-negative.

- Matrix Multiplication preserves non-negativity!!!

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If $A \geq 0$ (Element wise) with $A \in \mathbb{R}^{n \times n}$ and $z \geq 0$ with $z \in \mathbb{R}^{n}$, then $Az \geq 0$

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Given $A \in \mathbb{R}^{n \times n}$ with $A \geq 0$, then A is called regular if some $k \geq 1$, $A^k > 0$.

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Meaning of the Previous Definition

From a directed graph on nodes $1, ..., n$ with an arc from i to j whenever $A_{ij} \geq 0$ then

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A is regular if for some k there is a path of length k from every node to every other node.

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Perron-Frobenius Theorem for Regular Matrices

Theorem

Suppose $A \in \mathbb{R}^{n \times n}$ is nonnegative and regular, i.e., $A^k > 0$ for some k.

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Given P a stochastic matrix

- Let π a Perron-Frobenius right eigenvector of P with $\pi \geq 0$ and $1^T \pi = 1$.

Such that $P\pi = \pi$

- Then π corresponds to an invariant distribution or equilibrium distribution of the Markov chain for the eigenvalue 1.

Assume that

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Thus, a Simple Algorithms

We have a simple method

- Repeatedly apply $\boldsymbol{p}_{t+1} = \boldsymbol{P}\boldsymbol{p}_t$ until convergence to π.

This is a method called

- The Power Method

Naive and expensive but

- Stable!!!

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About 4000 years ago

Babylonians knew how to solve the following kind of systems

$$ax + by = c$$
$$dx + ey = f$$

As always the first steps in any field of knowledge tend to be slow

It is only after the death of Plato and Aristotle, that the Chinese (Nine Chapters of the Mathematical Art 200 B.C.) were able to solve 3×3 system.

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Not only that

The Matrix

Gauss defined implicitly the concept of a Matrix as linear transformations in his book "Disquisitions."

The Final Definition of Matrix

It was introduced by Cayley in two papers in 1850 and 1858 respectively, which allowed him to prove the important Cayley-Hamilton Theorem.

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Matrix can help to represent many things

They are important for many calculations as

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a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n =& b_2,\\
\cdots\cdots\cdots&\cdots\cdots\cdots\cdots\\
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It is clear

We would like to collect those linear equations in a compact structure that allows for simpler manipulation.

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Therefore, we have

For example

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \text{ and } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Using a little of notation

$$Ax = b$$

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$$A\boldsymbol{x} = \boldsymbol{b}$$

Furthermore

Many Times

- We use the concept of transpose a lot in many Linear Algebra Applications...

Definition

- The transpose of matrix A is an operation that flips the matrix over its diagonal. Formally, the i^{th} and j^{th} column element of A is the j^{th} row and i^{th} column element of A:

$$\left[A^T\right]_{ij} = [A]_{ji}$$

  - If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

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Solving Linear Equations

We have n equations in n unknowns

$$1x + 2y = 3$$
$$4x + 5y = 6$$

How do we solve this?

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Elimination

We can do the following

- Equation 2 - 4×Equation 1 $\Longrightarrow -3y = -6$

Back-Substitition

- $1x + 2 \times 2 = 3 \Longrightarrow x = -1$

Therefore

$$1\left(-1\right) + 2\left(2\right) = -3$$
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Determinants

There are a series of equations based on determinants

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \times 6 - 3 \times 4}{1 \times 5 - 2 \times 4} = \frac{-6}{-3} = 2$$

$$x = \frac{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{3 \times 5 - 2 \times 6}{1 \times 5 - 2 \times 4} = \frac{3}{-3} = -1$$

The Problem of Complexity

What if $n = 1000$

- Using the determinant rule is catastrophic

Given that the determinant $| |$

- You would have a million number to be used!!!

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- We favor the Gaussian Elimination method!!!

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- The idea of elimination is deceptively simple!!!
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To understand this method deeply

We need to understand

1. The Geometry of Linear Equations
2. Matrix Notation and Matrix Multiplication
3. Understand the Singular Cases
4. Number of Steps and Errors arising of the elimination

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Gaussian Elimination

Example

$$2x - y = 1$$
$$x + y = 5$$

We can look at that system by rows or by columns

- The first approach concentrates on the separate equations.

The Rows

- We have equations $2x - y = 1$ and $x + y = 5$.

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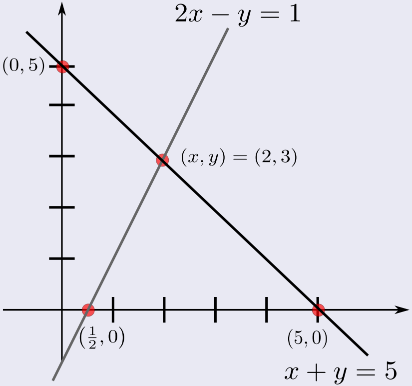


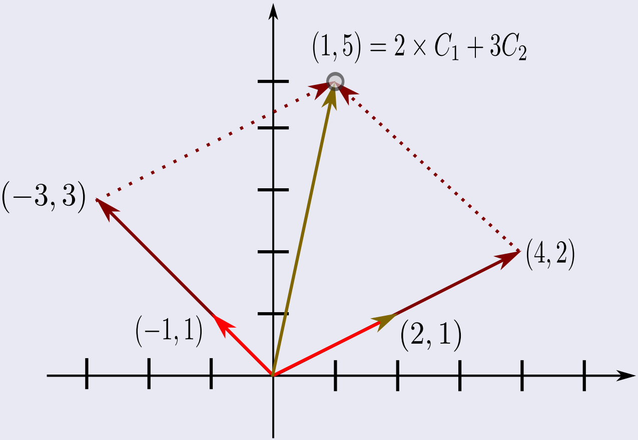
Now as columns of the linear systems

The two separate equations are really one vector equation

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$$

Thus, we have

As vectors



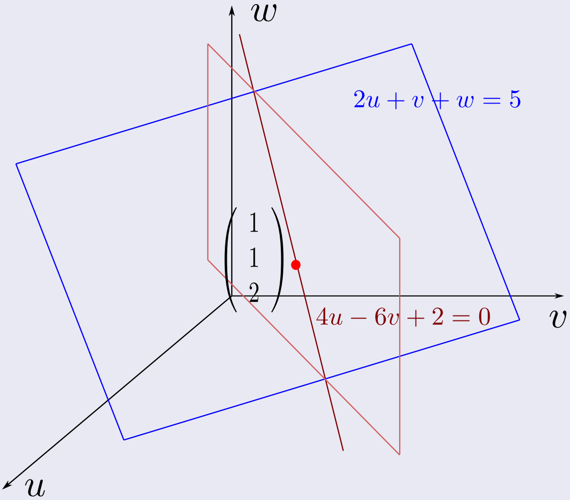
Now, if we move to $n = 3$

Three Planes

$$\begin{array}{rcrcrcr} 2u & + & v & + & w & = & 5 \\ 4u & - & 6v & & & = & -2 \\ -2u & + & 7v & + & 2w & = & 9 \end{array}$$

We have

The row picture



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What about column representation

Then, we have

$$u \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + v \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

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This allows to define

Vector addition

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \\ a_{31} \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + a_{13} \\ a_{21} + a_{22} + a_{23} \\ a_{31} + a_{32} + a_{33} \end{bmatrix}$$

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Further

Multiplication by scalars

$$x \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x \cdot a_1 \\ x \cdot a_2 \\ x \cdot a_3 \end{bmatrix}$$

Finally

Linear combination - combine the previous concepts

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{31} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} \\ x_1 a_{21} + x_2 a_{22} + x_3 a_{23} \\ x_1 a_{31} + x_2 a_{32} + x_3 a_{33} \end{bmatrix}$$

Our Goal, Beyond Two or Three Dimensions

We have

- With n equations in n unknowns, there are n planes in the row picture.
- There are n vectors in the column picture, plus a vector b on the right side.
- The equations ask for a linear combination of the n columns that equals b.

- Column picture: Combination of columns

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Outline

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Now, Imagine the following

What Happens with Three Planes in $\mathbb{R}^3$

- And they do not intersect

For example

- $2u + v + w = 5$
- $4u + 2v + 2w = 13$

Trying to do Gaussian Elimination

- It not allows to find a solution!!!

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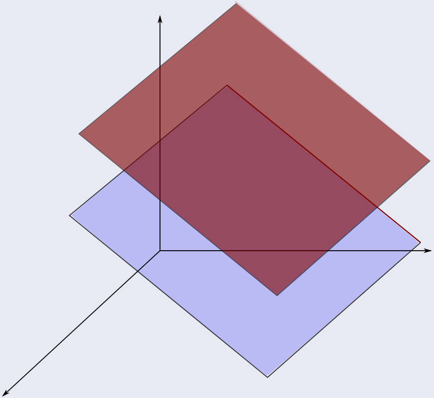
We could have have two parallel planes



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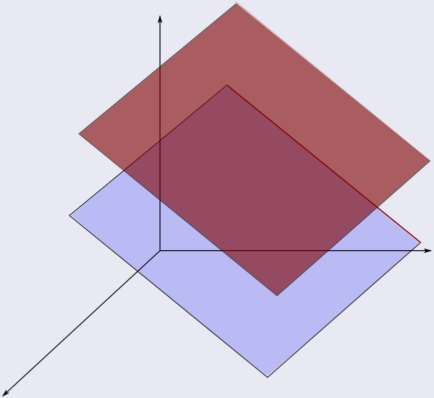

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For Example, No intersection

This corresponds to a singular system

$$\begin{aligned} u + v + w &= 2 \\ 2u + 3w &= 5 \\ 3u + v + 4w &= 6 \end{aligned}$$

- On the right side that fails: $2 + 5 \neq 6$
- Additionally 1 plus equation 2 minus equation 3 is the impossible statement $0 = 1$

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$$\begin{array}{ccccccc} u & + & v & + & w & = & 2 \\ 2u & & & + & 3w & = & 5 \\ 3u & + & v & + & 4w & = & 6 \end{array}$$

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What about an infinity set of solutions

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In such case

- The three planes have a whole line in common.

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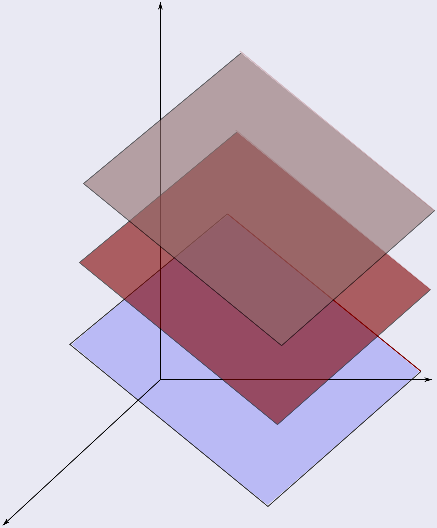
Finally

The extreme case is three parallel planes



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For special right sides, for example $b = (0, 0, 0)^T$

- The three parallel planes move over to become the same.
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Question

- What happens to the column picture when the system is singular?

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Outline

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We have an example

The System

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Applying Gaussian Elimination

First Step

- Subtracting multiples of the first equation from the other equations

For Example

- Subtract 2 times the first equation from the second.
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Therefore

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Pivot

We have that

- The coefficient 2 is the first pivot.

Something Notable

- Elimination is constantly dividing the pivot into the numbers underneath it.

Basically, if we put everything in terms of matrices

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 14 \end{pmatrix}$$

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Completing the process

Subtract 1 times the second equation from the third

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Know as a Triangular System

- Now substituting backwards, we can obtain our solutions:
  - This process is called back-substitution.

For example

- At the Board...

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A nice way to look at this substitution

We get our system to the bare minimum

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Outline

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Question

Important

- Under what circumstances could the process break down?

Very Simple

- If a zero appears in a pivot position.

The main problem

- We do not know whether a zero will appear until we try.

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We can try to cure this by exchanging rows!!!

Nonsingular - exchanging equations 2 and 3

$$\begin{bmatrix} 1 & 1 & 1 & - \\ 2 & 2 & 5 & - \\ 4 & 6 & 8 & - \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & - \\ 0 & 0 & 3 & - \\ 0 & 2 & 4 & - \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 & 1 & - \\ 0 & 2 & 4 & - \\ 0 & 0 & 3 & - \end{bmatrix}$$

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Outline

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How many separate arithmetical operations do we need?

Look at the Left Side of the Equations

- We have two types:
  - We divide by the pivot to know what multiple of the pivot equation is to be subtracted.

Therefore, we have a multiple l

- The terms in the pivot equation are multiplied by l,
  - Then subtracted from another equation.

If we have n operations to obtain the desired zeros of the other columns

- One to find l and the other to create the new entries.

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We have $n-1$ rows

- We need $n(n-1) = n^2 - n$ operations

This can be generalized

$$k \longmapsto k^2 - k$$

Then, we have for the Forward Elimination

$$\left(1^2 + 2^2 + \ldots + n^2\right) - (1 + 2 + \ldots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}$$

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If n is large

- The Left Side have a the following number of operations as $\frac{1}{3}n^3$.

Now for back substitution

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Total Number of Operations for the Back and Forward Steps

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Forward elimination also acts on the right-hand side

The Total Forward and Backward is responsible on the Right Side of

- $[(n-1)+(n-2)+...+1]+[1+2+...+n]=n^2$

Outline

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Thus

We start converting everything into a series of matrix multiplications

$$
\begin{aligned}
\mathbf{2}u + v + w &= 5 \\
-\mathbf{8}v - 2w &= -12 \\
\mathbf{1}w &= 2
\end{aligned}
$$

Then, we have a matrix representation!!!

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\begin{bmatrix}
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For This, we introduce the concept of Elementary Matrices

Definition

- In mathematics, an elementary matrix is a matrix which differs from the identity matrix by one single elementary row operation.

Row switching

A row within the matrix can be switched with another row

$$R_i \longleftrightarrow R_j$$

Example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Row multiplication

Each element in a row can be multiplied by a non-zero constant

$$kR_i \longrightarrow R_i \text{ where } k \neq 0$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Row addition

A row can be replaced by the sum of that row and a multiple of another row

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Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

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Outline

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We have then

Example

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

Then, we have

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

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Next

Second Row minus -4 first row

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Therefore, we have

Now

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Outline

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Vector Space V

Definition

A vector space V over the field K is a set of objects which can be added and multiplied by elements of K.

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Properties

We have then

1. Given elements $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ of V, we have $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$.
2. There is an element of V, denoted by O, such that $O+\boldsymbol{u}=\boldsymbol{u}+O=\boldsymbol{u}$ for all elements $\boldsymbol{u}$ of V.
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Outline

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Therefore, we have

We have the existence of a Zero Matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

The existence of $-A$

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Notation

First, $\boldsymbol{u} + (-\boldsymbol{v})$

As $\boldsymbol{u} - \boldsymbol{v}$.

For O

We will write sometimes 0.

The elements in the field K

They can receive the name of number or scalar.

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Outline

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Sub-spaces

Definition

Let V a vector space and $W \subseteq V$, thus W is a **subspace** if:

1. If $v, w \in W$, then $v + w \in W$.
2. If $v \in W$ and $c \in K$, then $cv \in W$.
3. The element $0 \in V$ is also an element of W.

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Example

Let $V \in \mathbb{R}^{d+1}$

W be the set of vectors in V whose last coordinate is equal to 0.

Do you remember?

- The augmented data Matrix X.

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Outline

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Some ways of recognizing Sub-spaces

Theorem

A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\boldsymbol{v}, \boldsymbol{w} \in W$ and each scalar $c \in K$ the vector $c\boldsymbol{v} + \boldsymbol{w} \in W$.

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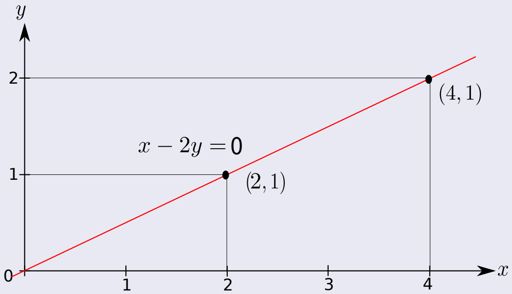
Example

For $\mathbb{R}^2$



Outline

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Linear Combinations

Definition

Let V an arbitrary vector space, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n \in V$ and $x_1, x_2, ..., x_n \in K$. Then, an expression like

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + ... + x_n\boldsymbol{v}_n \tag{6}$$

is called a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$.

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An Important Subspace

Something Notable

- Let W be the set of all linear combinations of subspace of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$. Then W is a subspace of V

Proof

- Take a look at the white board...

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Classic Examples

Endmember Representation in Hyperspectral Images

Look at the board

Geometric Representation of addition of forces in Physics

Look at the board!!

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An Interesting Example

Let $V = \mathbb{R}^d$

- Let $\boldsymbol{x}$ and $\boldsymbol{y} \in V$ such that

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \text{ and } \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix}$$

We define the dot product or scalar product

$$\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y} = \begin{pmatrix} x_1 & x_2 & \cdots & x_d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} = \sum_{i=1}^{d} x_i y_i$$

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Alternate Defintion

We can also define the dot product as

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sqrt{\sum_{i=1}^{d} x_i^2} \sqrt{\sum_{i=1}^{d} y_i^2} \cos \Theta$$

We have the following properties for this dot product

SP 1

- We have $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$

SP 2

- If $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z} \in V$

$$\boldsymbol{x} \cdot (\boldsymbol{y} + \boldsymbol{z}) = \boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{z} = (\boldsymbol{y} + \boldsymbol{z}) \cdot \boldsymbol{x}$$

SP 3

- if $k \in K$

$$(k\boldsymbol{x}) \cdot \boldsymbol{y} = k (\boldsymbol{x} \cdot \boldsymbol{y})$$
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Outline

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Properties and Definitions

Theorem

Let V be a vector space over the field K. The intersection of any collection of sub-spaces of V is a subspace of V.

- The sub-space spanned by S is defined as the intersection W of all sub-spaces of V which contains S.
- When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, we shall simply call W the sub-space spanned by the vectors $\alpha_1, \alpha_2, ..., \alpha_n$.

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We get the following Theorem

Theorem

The subspace spanned by $S \neq \emptyset$ is the set of all linear combinations of vectors in S.

Outline

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Linear Independence

Definition

Let V be a vector space over a field K, and let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n \in V$. We have that $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are linearly dependent over K if there are elements $a_1, a_2, ..., a_n \in K$ not all equal to 0 such that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + ... + a_n\boldsymbol{v}_n = O$$

Thus

Therefore, if there are not such numbers, then we say that $v_1, v_2, ..., v_n$ are linearly independent.

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Example!!!

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Basis

Definition

If elements $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ generate V and in addition are linearly independent, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ is called a **basis** of V. In other words the elements $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ form a basis of V.

Examples

The Classic Ones!!!

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Theorem

Let V be a vector space. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ be linearly independent elements of V. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be numbers. Suppose that we have

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n = y_1\boldsymbol{v}_1 + y_2\boldsymbol{v}_2 + \cdots + y_n\boldsymbol{v}_n \quad (7)$$

Then, $x_i = y_i$ for all $i = 1, \ldots, n$.

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For all $\boldsymbol{v} \in V$, $\boldsymbol{v} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_n\boldsymbol{v}_n$.

Thus, this n-tuple is uniquely determined by v

We will call $(x_1, x_2, \ldots, x_n)$ as the coordinates of v with respect to the basis.

The $n-$tuple $X = (x_1, x_2, \ldots, x_n)$

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Properties of a Basis

Theorem - (Limit in the size of the basis)

Let V be a vector space over a field K with a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m\}$. Let $\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n$ be elements of V, and assume that $n > m$. Then $\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n$ are linearly dependent.

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Some Basic Definitions

We will define the dimension of a vector space V over K

As the number of elements in the basis.

- Denoted by $\dim_K V$, or simply $\dim V$

Therefore

A vector space with a basis consisting of a finite number of elements, or the zero vector space, is called a **finite dimensional.**

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Maximal Set of Linearly Independent Elements

Theorem

Let V be a vector space, and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ a maximal set of linearly independent elements of V. Then, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n\}$ is a basis of V.

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Let V be a vector space of dimension n, and let $v_1, v_2, ..., v_n$ be linearly independent elements of V. Then, $v_1, v_2, ..., v_n$ constitutes a basis of V.

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Let V be a vector space and let W be a subspace. If $\dim W = \dim V$ then $V = W$.

Corollary

Let V be a vector space of dimension n. Let r be a positive integer with $r < n$, and let $v_1, v_2, ..., v_r$ be linearly independent elements of V. Then one can find elements $v_{r+1}, v_{r+2}, ..., v_n$ such that $\{v_1, v_2, ..., v_n\}$ is a basis of V.

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Finally

Theorem

Let V be a vector space having a basis consisting of n elements. Let W be a subspace which does not consist of O alone. Then W has a basis, and the dimension of W is $\leq n$.

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Outline

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We have

Given a Matrix A of $m \times n$

- And it has been reduced to a a Echelon or Reduced Version...

It is possible to find the subspaces associated A

- Here, we introduce the concept of rank

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- the rank of a matrix A is the dimension of the vector space generated (or spanned) by its columns.

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Then, we have the following subspaces

1. The **column space** of A is denoted by $C(A)$ is a subspace of $\mathbb{R}^m$.
   - Its dimension is the rank r.
2. The **null space** of A is denoted by $N(A)$ is a subspace of $\mathbb{R}^n$.
   - It contains all vectors y such that $Ay = 0$.
   - Its dimension is $n - r$.
3. The **row space** of A is the column space of A^T $(n \times m)$, a subspace of $\mathbb{R}^n$.
   - It is $C\left(A^T\right)$, and it is spanned by the rows of A.
   - Its dimension is also r.
4. The left **null space** of A is the nullspace of A^T and subspace of $\mathbb{R}^m$.
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Now

If A is an $m \times n$ matrix

- We have a host spaces that contains these fundamental subspaces.

First

- The nullspace $N(A)$ and row space $C\left(A^T\right)$ are subspaces of $\mathbb{R}^n$.

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Example

If we have

$$A = U = R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the Column space is the line through

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The Row space is the line through

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We have that $N(A)$ contains

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What about $N\left(A^T\right)$?

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We have

Our Basic Matrix

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}$$

Therefore, we have the Reduced Matrix After Gaussian Elimination

$$U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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- For an echelon matrix like U, the row space is clear.

It contains all combinations of the rows

- However, the third Row does not adds anything!!!

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- A similar rule applies to every echelon matrix U, with r pivots and r nonzero rows.

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- The row space of A has the same dimension r as the row space of U, and it has the same bases.

Why?

- The reason is that each elementary operation leaves the row space unchanged.

Further

- The rows in U are combinations of the original rows in A.
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After all

$$Ax = 0 \longrightarrow Ux = 0$$

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Only r of the equations $A\boldsymbol{x} = 0$ are independent.

- We can see that...

Thus, choosing the $n - r$ "special solutions" to $A\boldsymbol{x} = 0$

- The null space $N(A)$ has dimension $n - r$.

The "special solutions" are a basis

- Each free variable is given the value 1, while the other free variables are 0.

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Then $Ax = 0$ or $Ux = 0$

Using Back Substitution we can obtain the variables

$$U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is exactly the way we have been solving $Ux = 0$

- The basic example above has pivots in columns 1 and 3.

Remember

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Therefore its free variables are the second and fourth

$$\text{Special Solutions } \boldsymbol{x}_1 = \begin{pmatrix} -3 \\ 1 \leftarrow \\ 0 \\ 0 \leftarrow \end{pmatrix}, \boldsymbol{x}_2 = \begin{pmatrix} 1 \\ 0 \leftarrow \\ -1 \\ 1 \leftarrow \end{pmatrix}$$

Any combination $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2$

- It has c_1 as its second component, and c_2 as its fourth component.

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The only way to have $c_1\boldsymbol{x}_1 + c_2\boldsymbol{x}_2 = 0$

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They are basis for the null space because

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Additionally

The null space is also called the kernel of A

- Its dimension $n - r$ is the nullity.

Now, we have

The column space is sometimes called the range

- After all you can define linear functions using matrices

$$f_A : \mathbb{R}^n \to \mathbb{R}^m, \text{ with } f_A(\boldsymbol{x}) = A\boldsymbol{x}$$

Then

1. Its domain consists of all x in $\mathbb{R}^n$.
2. Its range is all possible vectors $f_A(x) = Ax$.

Our problem is to find bases for the column spaces of U and A

- Those spaces are different, but their dimensions are the same

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Example

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$$U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}$$

- Every other column is a combination of these two!!!

- The fourth column equals (column 3) − (column 1)

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Further

The same null space is telling us those dependencies

- The reason is this

$$A\boldsymbol{x} = 0 \Leftrightarrow U\boldsymbol{x} = 0$$

Every linear dependence $A\boldsymbol{x} = 0$ among the columns of A

- It is matched by a dependence $U\boldsymbol{x} = 0$ among the columns of U, with exactly the same coefficients.

Theorem

If a set of columns of A is independent, then so are the corresponding columns of U, and vice versa.

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Now

To find a basis for the column space $C(A)$, we use what is already done for U

- The r columns containing pivots are a basis for the column space of U.

Something Notable

- We will pick those same r columns in A.

Observation

- The dimension of the column space $C(A)$ equals the rank r, which also equals the dimension of the row space.

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- The number of independent columns equals the number of independent rows.

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- A basis for $C(A)$ is formed by the r columns of A that correspond, in U, to the columns containing pivots.

It also says something about square matrices

- If the rows of a square matrix are linearly independent, then so are the columns (and vice versa).

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How?

both the row and column spaces of U have dimension $r = 3$

$$U = \begin{pmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We claim that U also has three independent columns, and no more

- We notice, the columns have only three nonzero components.

If we can show that the pivot columns are linearly independent

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Back Substitution

Look at the Board...

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Since

- $A\boldsymbol{x} = 0$ if and only if $U\boldsymbol{x} = 0$.

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We can call this as

The null space of A^T

- If A is an $m \times n$ matrix $\Longrightarrow$ A^T is a $n \times m$ matrix.

Thus, its null space is a subspace of $\mathbb{R}^m$

We have:

1. $y^T A = 0$
2. $A^T y = 0$

Example

$$y^T A = [y_1, \cdots, y_m] A = [0 \cdots 0]$$

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What is the dimension of the Null Space $N\left(A^T\right)$

For any matrix

- **The number of pivot variables plus the number of free variables must match the total number of columns.**

In other words

dimension of $C(A)$+ dimension of $N(A)$= number of columns

This law applies to A^T which has m columns, then

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The left nullspace $N\left(A^T\right)$

- It has dimension $m - r$

Outline

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Fundamental Theorem of Linear Algebra

Given A

1. $C(A)$ = column space of A; dimension r
2. $N(A)$ = null space space of A; dimension $n - r$
3. $C\left(A^T\right)$ = row space of A; dimension r
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We have the following situation

We know that if A has a left-inverse $(BA = I)$ and a right-inverse $(AC = I)$

$$B = BI = B(AC) = (BA)C = C$$

Now, from the rank of a matrix

- We can decide if the matrix has these inverses.

Properties

- An inverse exists only when the rank is as large as possible.

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First

The rank always satisfies $r \leq m$ and also $r \leq n$

- Thus, an m by n matrix cannot have more than m independent rows or n independent columns.

- When $r = n$ there is a left inverse, and the solution (if it exists) is unique.

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EXISTENCE

Full row rank $r = m$

- Then, $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution $\boldsymbol{x}$ for every b if and only if the columns span $\mathbb{R}^m$

Therefore

- Then A has a right-inverse C such that $AC = I_m (m \times m)$.

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UNIQUENESS

Full column rank $r = n$

- $A\boldsymbol{x} = b$ has at most one solution x for every b if and only if the columns are linearly independent.

Then A has an $n \times m$ left-inverse B

$$BA = I_n$$

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Existence Case

Once possible solution is $\boldsymbol{x} = C\boldsymbol{b}$

- Then, $A\boldsymbol{x} = AC\boldsymbol{b} = \boldsymbol{b}$

Something Notable

- But there will be other solutions if there are other right-inverses.

Finally

- The number of solutions when the columns span $\mathbb{R}^m$ is 1 to ∞.

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In the uniqueness case

If there is a solution to $A\boldsymbol{x} = \boldsymbol{b}$

- It has to be $\boldsymbol{x} = BA\boldsymbol{x} = B\boldsymbol{b}$

But there may be no solution

- The number of solutions is 0 or 1.

There are simple formulas for the best left and right inverses

$$B = \left(A^T A\right)^{-1} A \text{ and } C = A^T \left(AA^T\right)^{-1}$$

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Therefore

Certainly $BA = I$ and $AC = I$

- Look at the Board

Something Notable

- What is not so certain is that $A^T A$ and AA^T are actually invertible.

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- A rectangular matrix cannot have both existence and uniqueness.

Why?

- If m is different from n, we cannot have $r = m$ and $r = n$.

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A square matrix is the opposite

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- If $m = n$, we cannot have one property without the other.

Furthermore

- A square matrix has a left-inverse if and only if it has a right-inverse.

Another Condition

- The condition for invertibility is full rank: $r = m = n$.

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Thus

Each of these conditions is a necessary and sufficient test

1. The columns span $\mathbb{R}^n$, so $Ax = b$ has at least one solution for every b.
2. The columns are independent, so $Ax = 0$ has only the solution $x = 0$.

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We have a longer list

The following conditions are equivalent

1. The rows of A span $\mathbb{R}^n$.
2. The rows are linearly independent.
3. Elimination can be completed: $PA = LDU$, with all n pivots.
4. The determinant of A is not zero.
5. Zero is not an eigenvalue of A.
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